HOMEWORK ASSIGNMENT #4, MATH 253

1. Prove that the following differential equations are satisfied by the given functions:

   (a) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$, where $u = (x^2 + y^2 + z^2)^{-1/2}$.

   (b) $x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = -2w$, where $w = \left(x^2 + y^2 + z^2\right)^{-1}$.

2. Show that the function $u = t^{-1}e^{-(x^2+y^2)/4t}$ satisfies the two dimensional heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$.

3. (a) Find an equation of the tangent plane to the surface $x^2 + y^2 + z^2 = 9$ at the point $(2, 2, 1)$.

   (b) At what points (x, y, z) on the surface in part (a) are the tangent planes parallel to $2x + 2y + z = 1$?

4. Find the points on the ellipsoid $x^2 + 2y^2 + 3z^2 = 1$ where the tangent plane is parallel to the plane $3x - y + 3z = 1$.

5. (a) Find an equation for the tangent line to the curve of intersection of the surfaces

   $$x^2 + y^2 + z^2 = 9 \text{ and } 4x^2 + 4y^2 - 5z^2 = 0 \text{ at the point } (1, 2, 2).$$

   (b) Find the radius of the sphere whose center is $(-1, -1, 0)$ and which is tangent to the plane $x + y + z = 1$.

6. Find the point(s) on the surface $z = xy$ that are nearest to the point $(0, 0, 2)$.

7. Let $f(x, y, z)$ be the function defined by $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$. Determine an equation for the normal line of the surface $f(x, y, z) = 3$ at the point $(-1, 2, 2)$.

8. Let $f(x, y, z) = \frac{xy}{z}$. Measurements are made and it is found that $x = 10, y = 10, z = 2$. If the maximum error made in each measurement is 1% find the approximate percentage error made in computing the value of $f(10, 10, 2)$.

9. Find all points on the surface given by

   $$(x - y)^2 + (x + y)^2 + 3z^2 = 1$$

   where the tangent plane is perpendicular to the plane $2x - 2y = 13$.

10. Find all points at which the direction of fastest change of $f(x, y) = x^2 + y^2 - 2x - 4y$ is $\vec{i} + \vec{j}$.

11. The surface $x^4 + y^4 + z^4 + xyz = 17$ passes through $(0, 1, 2)$, and near this point the surface determines x as a function, $x = F(y, z)$, of y and z.

    (a) Find F_y and F_z at $(x, y, z) = (0, 1, 2)$.

    (b) Use the tangent plane approximation (otherwise known as linear, first order or differential approximation) to find the approximate value of x (near 0) such that $(x, 1.01, 1.98)$ lies on the surface.

12. Let $f(x, y)$ be a differentiable function, and let $u = x + y$ and $v = x - y$. Find a constant α such that
$$(f_x)^2 + (f_y)^2 = \alpha((f_u)^2 + (f_v)^2).$$

13. Find the directional derivative $D_{\vec{u}}f$ at the given point in the direction indicated by the angle

    (a) $f(x, y) = \sqrt{5x - 4y},\ (2, 1),\ \theta = -\pi/6.$

    (b) $f(x, y) = x\sin(xy),\ (2, 0),\ \theta = \pi/3.$

14. Compute the directional derivatives $D_{\vec{u}}f$, where:

    (a) $f(x, y) = \ln(x^2 + y^2)$, $\vec{u}$ is the unit vector pointing from $(0, 0)$ to $(1, 2)$.

    (b) $f(x, y, z) = \dfrac{1}{\sqrt{x^2 + 2y^2 + 3z^2}}$, $\vec{u} =< 1/\sqrt{2}, 1/\sqrt{2}, 0 >$.

15. Find all points (x, y, z) such that $D_{\vec{u}}f(x, y, z) = 0$, where $\vec{u} =< a, b, c >$ is a unit vector and $f(x, y, z) = \sqrt{\alpha x^2 + \beta y^2 + \gamma z^2}$.

16. Compute the cosine of the angle between the gradient ∇f and the positive direction of the z-axis, where $f(x, y, z) = x^2 + y^2 + z^2$.

17. The temperature at a point (x, y, z) is given by $T(x, y, z) = 200e^{-x^2-3y^2-9z^2}$.

    (a) Find the rate of change of temperature at the point $P(2, -1, 2)$ in the direction towards the point $(3, -3, 3)$.

    (b) In which direction does the temperature increase the fastest at P?

    (c) Find the maximum rate of increase at P.

SOLUTIONS TO HOMEWORK ASSIGNMENT #4, MATH 253

1. Prove that the following differential equations are satisfied by the given functions:

   (a) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$, where $u = (x^2 + y^2 + z^2)^{-1/2}$.

   (b) $x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = -2w$, where $w = \left(x^2 + y^2 + z^2\right)^{-1}$.

   Solution:

   (a) $\frac{\partial u}{\partial x} = -x(x^2+y^2+z^2)^{-3/2}$ and $\frac{\partial^2 u}{\partial x^2} = -(x^2+y^2+z^2)^{-3/2} + 3x^2(x^2+y^2+z^2)^{-5/2}$. By symmetry we see that

$$\begin{aligned} \frac{\partial^2 u}{\partial y^2} &= -(x^2+y^2+z^2)^{-3/2} + 3y^2(x^2+y^2+z^2)^{-5/2} \\ \frac{\partial^2 u}{\partial z^2} &= -(x^2+y^2+z^2)^{-3/2} + 3z^2(x^2+y^2+z^2)^{-5/2}. \end{aligned}$$

   Adding up clearly gives 0.

   (b) $x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = -2x^2(x^2+y^2+z^2)^{-2} - 2y^2(x^2+y^2+z^2)^{-2} - 2z^2(x^2+y^2+z^2)^{-2} = -2w$.

2. Show that the function $u = t^{-1}e^{-(x^2+y^2)/4t}$ satisfies the two dimensional heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$.

   Solution:

$$\begin{aligned} \frac{\partial u}{\partial t} &= -t^{-2}e^{-(x^2+y^2)/4t} + \frac{x^2+y^2}{4t^3}e^{-(x^2+y^2)/4t} \\ \frac{\partial u}{\partial x} &= -\frac{x}{2t^2}e^{-(x^2+y^2)/4t}, \quad \frac{\partial u}{\partial y} = -\frac{y}{2t^2}e^{-(x^2+y^2)/4t} \\ \frac{\partial^2 u}{\partial x^2} &= -\frac{1}{2t^2}e^{-(x^2+y^2)/4t} + \frac{x^2}{4t^3}e^{-(x^2+y^2)/4t} \\ \frac{\partial^2 u}{\partial y^2} &= -\frac{1}{2t^2}e^{-(x^2+y^2)/4t} + \frac{y^2}{4t^3}e^{-(x^2+y^2)/4t} \end{aligned}$$

   It is now clear that $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$.

3. (a) Find an equation of the tangent plane to the surface $x^2 + y^2 + z^2 = 9$ at the point $(2, 2, 1)$.

(b) At what points (x, y, z) on the surface in part (a) are the tangent planes parallel to $2x + 2y + z = 1$?

Solution:

(a) If $f(x, y, z) = x^2 + y^2 + z^2$ then a normal of the surface $x^2 + y^2 + z^2 = 9$ at $(2, 2, 1)$ is given by the gradient $\boldsymbol{\nabla} f(x, y, z)|_{(2,2,1)} = (2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k})|_{\{x=2,y=2,z=1\}} = 4\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$. Actually we will take the normal to be $\mathbf{n} = 2\mathbf{i}+2\mathbf{j}+\mathbf{k}$. The extra factor 2 is not needed. Thus the equation of the tangent plane to the surface $x^2 + y^2 + z^2 = 9$ at the point $(2, 2, 1)$ is $2(x - 2) + 2(y - 2) + (z - 1) = 0$, that is $2x + 2y + z = 9$.

(b) The point here is that the family of planes $2x + 2y + z = \lambda$ forms a complete family of parallel planes as λ varies, $-\infty < \lambda < \infty$. Thus the points on the sphere $x^2 + y^2 + z^2 = 9$ where the tangent plane is parallel to $2x + 2y + z = 1$ are $\pm(2, 2, 1)$. From part (a) we see that one of the points is $(2, 2, 1)$. The diametrically opposite point $-(2, 2, 1)$ is the only other point. This follows from the geometry of the sphere.

4. Find the points on the ellipsoid $x^2 + 2y^2 + 3z^2 = 1$ where the tangent plane is parallel to the plane $3x - y + 3z = 1$.

   Solution:

   We want (x, y, z) such that $x^2 + 2y^2 + 3z^2 = 1$ and $< 2x, 4y, 6z >= \lambda < 3, -1, 3 >$, for some λ, that is $x = 3\lambda/2, y = -\lambda/4, z = \lambda/2$. Thus we must have

$$x^2 + 2y^2 + 3z^2 = (9/4 + 1/8 + 3/4)\lambda^2 = 1 \Longrightarrow \lambda = \pm\frac{2\sqrt{2}}{5}.$$

   Therefore $(x, y, z) = \pm\left(\frac{3\sqrt{2}}{5}, -\frac{\sqrt{2}}{10}, \frac{\sqrt{2}}{5}\right)$.

5. (a) Find an equation for the tangent line to the curve of intersection of the surfaces

$$x^2 + y^2 + z^2 = 9 \text{ and } 4x^2 + 4y^2 - 5z^2 = 0 \text{ at the point } (1, 2, 2).$$

   (b) Find the radius of the sphere whose center is $(-1, -1, 0)$ and which is tangent to the plane $x + y + z = 1$.

   Solution:

   (a) By taking gradients (up to constant multiples) we see that the respective normals at $(1, 2, 2)$ are $\mathbf{n_1} = \mathbf{i}+2\mathbf{j}+2\mathbf{k}$ and $\mathbf{n_2} = 2\mathbf{i}+4\mathbf{j}-5\mathbf{k}$. Thus a direction vector at $(1, 2, 2)$ for the curve of intersection is $\mathbf{n} = \mathbf{n_1}\times\mathbf{n_2} = -18\mathbf{i} + 9\mathbf{j}$. Removing a factor of 9 we see that a direction vector is $\mathbf{v} = -2\mathbf{i} + \mathbf{j}$, and therefore the equation of the tangent line is $x = 1 - 2t, y = 2 + t, z = 2$.

   (b) The sphere will have the equation $(x + 1)^2 + (y + 1)^2 + z^2 = r^2$ for some r. In order for this sphere to be tangent to the plane $x + y + z = 1$ it is necessary that the "radius vector" be proportional to the normal vector to the plane, that is $(x+1, y+1, z) = \lambda(1,1,1)$ for some λ. But we must also have $x+y+z=1$ and therefore $\lambda = 1$. It follows that $r = \sqrt{3}$.

6. Find the point(s) on the surface $z = xy$ that are nearest to the point $(0,0,2)$.

   Solution:

   Let $F(x,y,z) = z - xy$. Thus the surface is the level surface $F(x,y,z) = 0$. We want to find all points (x,y,z) on the surface where the gradient $\nabla F(x,y,z)$ is parallel to the vector pointing from $(0,0,2)$ to (x,y,z). Therefore

$$x = -\lambda y, y = -\lambda x, z - 2 = \lambda, \text{ and } z = xy.$$

   The solutions are

$$(x,y,z) = (0,0,0), \lambda = -2; (x,y,z) = (1,1,1), \lambda = -1; (x,y,z) = (-1,-1,1), \lambda = -1.$$

   Clearly $\exists$ closest point(s). They are $(1,1,1)$ and $(-1,-1,1)$.

7. Let $f(x,y,z)$ be the function defined by $f(x,y,z) = \sqrt{x^2+y^2+z^2}$. Determine an equation for the normal line of the surface $f(x,y,z) = 3$ at the point $(-1,2,2)$.

   Solution:

   A normal to the surface $f(x,y,z) = 3$ at $(-1,2,2)$ is $\vec{n} = < -1/3, 2/3, 2/3 >$. Thus an equation for the normal line is

$$x = -1 - \frac{1}{3}t, y = 2 + \frac{2}{3}t, z = 2 + \frac{2}{3}t, -\infty < t < \infty.$$

8. Let $f(x,y,z) = \dfrac{xy}{z}$. Measurements are made and it is found that $x = 10, y = 10, z = 2$. If the maximum error made in each measurement is 1% find the approximate percentage error made in computing the value of $f(10,10,2)$.

   Solution: The calculated value is $f(10,10,2) = 50$, with errors

$$-0.1 \le \Delta x \le 0.1, \ -0.1 \le \Delta y \le 0.1 \text{ and } -0.02 \le \Delta z \le 0.02.$$

   The approximate error made is

$$f(10+\Delta x, 10+\Delta y, 2+\Delta z) - f(10,10,2) \approx \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}\Delta x$$
$$= 5\Delta x + 5\Delta y - 25\Delta z.$$

   Then we have $-1.5 \le 5\Delta x + 5\Delta y - 25\Delta z \le 1.5$, and so the approximate percentage error is $\dfrac{1.5}{50} \times 100\% = 3\%$.

9. Find all points on the surface given by

$$(x-y)^2+(x+y)^2+3z^2=1$$

where the tangent plane is perpendicular to the plane $2x-2y=13$.

Solution:

A normal to the surface $(x-y)^2+(x+y)^2+3z^2=1$ is $\vec{n}=<4x,4y,6z>$. Thus we want to solve simultaneously the equations $(x-y)^2+(x+y)^2+3z^2=1$ and $<4x,4y,6z>\cdot<2,-2,0>=0$. Thus the points are (x,x,z), where x,z lie on the ellipse $4x^2+3z^2=1$.

10. Find all points at which the direction of fastest change of $f(x,y)=x^2+y^2-2x-4y$ is $\vec{i}+\vec{j}$.

Solution:

The direction of fastest change is in the direction of $\nabla f=<2x-2,2y-4>$. Therefore we want $\nabla f=<2x-2,2y-4>=(\lambda,\lambda)$, that is $2x-2=2y-4$. Therefore the points are $(x,y)=(x,x+1),-\infty<x<\infty$.

11. The surface $x^4+y^4+z^4+xyz=17$ passes through $(0,1,2)$, and near this point the surface determines x as a function, $x=F(y,z)$, of y and z.

(a) Find F_y and F_z at $(x,y,z)=(0,1,2)$.

(b) Use the tangent plane approximation (otherwise known as linear, first order or differential approximation) to find the approximate value of x (near 0) such that $(x,1.01,1.98)$ lies on the surface.

Solution:

(a) To find $\dfrac{\partial x}{\partial y}$ and $\dfrac{\partial x}{\partial z}$ at $(y,z)=(1,2)$ we differentiate the equation

$$x^4+y^4+z^4+xyz=17 \text{ with respect to } y,z;$$

then put $x=0,y=1,z=2$, and finally solve for $\dfrac{\partial x}{\partial y}$ and $\dfrac{\partial x}{\partial z}$:

$$\frac{\partial}{\partial y}(x^4+y^4+z^4+xyz)=0 \implies 4x^3\frac{\partial x}{\partial y}+4y^3+\frac{\partial x}{\partial y}yz+xz=0\implies\frac{\partial x}{\partial y}=-2$$
$$\frac{\partial}{\partial z}(x^4+y^4+z^4+xyz)=0 \implies 4x^3\frac{\partial x}{\partial z}+4z^3+\frac{\partial x}{\partial z}yz+xy=0\implies\frac{\partial x}{\partial z}=-16$$

(b) The tangent plane approximation is

$$F(y+\Delta y,z+\Delta z)\approx F(y,z)+\frac{\partial F}{\partial y}\Delta y+\frac{\partial F}{\partial z}\Delta z.$$

In this case $F(1,2)=0$ and thus $F(1.01,1.98)\approx 0-2\times 0.01+16\times 0.02=0.3$.

12. Let $f(x,y)$ be a differentiable function, and let $u = x+y$ and $v = x-y$. Find a constant α such that

$$(f_x)^2 + (f_y)^2 = \alpha((f_u)^2 + (f_v)^2).$$

Solution: By the chain rule

$$(f_x)^2 + (f_y)^2 = (f_u + f_v)^2 + (f_u - f_v)^2 = 2((f_u)^2 + (f_v)^2). \text{ Thus } \alpha = 2.$$

13. Find the directional derivative $D_{\vec{u}}f$ at the given point in the direction indicated by the angle

(a) $f(x,y) = \sqrt{5x-4y},\ (2,1),\ \theta = -\pi/6.$

(b) $f(x,y) = x\sin(xy),\ (2,0),\ \theta = \pi/3.$

Solution:

(a) $D_{\vec{u}}f = \left(\frac{5}{2\sqrt{6}}\vec{i} - \frac{2}{\sqrt{6}\vec{j}}\right) \cdot \left(\frac{\sqrt{3}}{2}\vec{i} - \frac{1}{2}\vec{j}\right) = \frac{5}{4\sqrt{2}} + \frac{1}{\sqrt{6}}.$

(b) $D_{\vec{u}}f = 4\vec{j} \cdot \left(\frac{1}{2}\vec{i} + \frac{\sqrt{3}}{2}\vec{J}\right) = 2\sqrt{3}.$

14. Compute the directional derivatives $D_{\vec{u}}f$, where:

(a) $f(x,y) = \ln(x^2+y^2)$, $\vec{u}$ is the unit vector pointing from $(0,0)$ to $(1,2)$.

(b) $f(x,y,z) = \frac{1}{\sqrt{x^2+2y^2+3z^2}},\ \vec{u} =< 1/\sqrt{2}, 1/\sqrt{2}, 0 > .$

Solution:

(a) $D_{\vec{u}}f = \frac{2x}{x^2+y^2}\frac{1}{\sqrt{5}} + \frac{2y}{x^2+y^2}\frac{2}{\sqrt{5}} = \frac{1}{\sqrt{5}}\frac{2x+4y}{x^2+y^2}.$

(b)

$$\begin{aligned} D_{\vec{u}}f &= -\frac{1}{\sqrt{2}}\frac{x}{(x^2+2y^2+3z^2)^{3/2}} - \frac{1}{\sqrt{2}}\frac{2y}{(x^2+2y^2+3z^2)^{3/2}} \\ &= -\frac{1}{\sqrt{2}}\frac{x+2y}{(x^2+2y^2+3z^2)^{3/2}} \end{aligned}$$

15. Find all points (x,y,z) such that $D_{\vec{u}}f(x,y,z) = 0$, where $\vec{u} =< a,b,c >$ is a unit vector and $f(x,y,z) = \sqrt{\alpha x^2 + \beta y^2 + \gamma z^2}$.

Solution:

$D_{\vec{u}}f = \frac{a\alpha x + b\beta y + c\gamma z}{\sqrt{\alpha x^2 + \beta y^2 + \gamma z^2}} = 0 \Longleftrightarrow a\alpha x + b\beta y + c\gamma z = 0.$

16. Compute the cosine of the angle between the gradient ∇f and the positive direction of the z-axis, where $f(x,y,z) = x^2+y^2+z^2$.

Solution: For $\cos\theta = \frac{(\nabla f)\cdot\vec{k}}{|\nabla f|} = \frac{z}{\sqrt{x^2+y^2+z^2}}.$

17. The temperature at a point (x, y, z) is given by $T(x, y, z) = 200e^{-x^2-3y^2-9z^2}$.

(a) Find the rate of change of temperature at the point $P(2, -1, 2)$ in the direction towards the point $(3, -3, 3)$.

(b) In which direction does the temperature increase the fastest at P?

(c) Find the maximum rate of increase at P.

Solution:

(a)

$$\begin{aligned} D_{\vec{u}}T &= \nabla T \cdot \frac{1}{\sqrt{6}} < 1, -2, 1 >= -\frac{400}{e^{x^2+3y^2+9z^2}} < x, 3y, 9z > \cdot \frac{1}{\sqrt{6}} < 1, -2, 1 > \\ &= -\frac{400}{e^{43}\sqrt{6}}(x - 6y + 9z) = -\frac{400 \times 26}{e^{43}\sqrt{6}} = -\frac{-10400}{e^{43}\sqrt{6}}. \end{aligned}$$

(b) In the direction of the gradient. A unit vector pointing in the direction of ∇T at the point $(2, -1, 2)$ is $\vec{u} = -\frac{1}{\sqrt{337}} < 2, -3, 18 >$.

(c) The maximum rate of increase of $T(x, y, z)$ at the point $(2, -1, 2)$ is

$$|\nabla T| = \frac{400 \times \sqrt{337}}{e^{43}}.$$