Midterm 1 October 12, 2016 Duration: 50 minutes

This test has 4 ***questions*** *on* ***5 pages****, each worth 10 points, for a total of 40 points.*

- Read all the questions carefully before starting to work.
- Give complete arguments and explanations for all your calculations. Answers without justifications will not be marked, except question #3 where the answer alone is sufficient.
- Continue on the closest blank page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **No aids of any kind are allowed**, including: documents, cheat sheets, electronic devices of any kind (including calculators, phones, etc.)

First Name: Solutions Last Name: ______

Student-No: ______ Section: ______

Signature: ______

Question:	1	2	3	4	Total
Points:	10	10	10	10	40
Score:					

2 marks 1. (a) Consider the three points $A = (0, 1, 1), B = (2, 1, 0), C = (3, -1, 2)$. Find the cosine of the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

Answer: $\sqrt{\frac{5}{14}}$

Solution: The angle θ between the vectors $\overrightarrow{AB} = \langle 2, 0, -1\rangle$ and $\overrightarrow{AC} = \langle 3, -2, 1\rangle$ is given by

$$\cos\theta = \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|} = \frac{5}{\sqrt{5}\sqrt{14}} = \sqrt{\frac{5}{14}}.$$

2 marks (b) Continuing from previous part, find a normal vector to the plane containing the points A, B and C.

Answer: $\langle -2, -5, -4\rangle$

Solution: A normal vector to the desired plane is given by

$$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & -1 \\ 3 & -2 & 1 \end{vmatrix} = \langle -2, -5, -4\rangle.$$

3 marks (c) Find the equation of the line (in symmetric form) which is at the intersection of the planes $3x - 4y + 2z = -7$ and $3x - 2y + 4z = -5$

Answer: $\frac{x+1}{-2} = \frac{y-1}{-1} = \frac{z}{1}$

Solution: Subtract the two equations to eliminate x, we obtain $y + z = 1$. Eliminating y yields $x + 2z = -1$. Expressing z in terms of x and y, we obtain

$$\frac{x+1}{-2} = \frac{y-1}{-1} = \frac{z}{1}$$

3 marks (d) Find the equations of all the planes that are at distance 1 unit from the plane $x + y - z = 1$.

Answer: $x + y - z = 1 \pm \sqrt{3}$

Solution: Two planes separated by a positive distance must be parallel to each other. Therefore the equation of the plane is of the form

$$x + y - z = k.$$

The normal to the planes is $\vec{n} = \langle 1, 1, -1\rangle$. The points $A = (1, 0, 0)$ and $B = (k, 0, 0)$ are on the planes $x+y-z = 1$ and $x+y-z = k$ respectively. Therefore the distance between the planes is given by the scalar projection of $\overrightarrow{AB}$ on $\vec{n}$,

$$\frac{\left|\overrightarrow{AB}\cdot\vec{n}\right|}{|\vec{n}|} = \frac{|k-1|}{\sqrt{3}} = 1.$$

This implies $k = 1 \pm \sqrt{3}$.

4 marks

2. (a) Let $f(x,y) = ye^{2x-y} + 4x\sin(y) + 3x^2$. Compute the partial derivatives f_x and f_y.

Answer: $f_x = 2ye^{2x-y} + 4\sin(y) + 6x$,
$f_y = (1-y)e^{2x-y} + 4x\cos(y)$.

Solution: Partial differentiation simply gives

$$f_x(x,y) = 2ye^{2x-y} + 4\sin(y) + 6x,$$
$$f_y(x,y) = e^{2x-y} - ye^{2x-y} + 4x\cos(y).$$

6 marks

(b) Find all values of the constant c such that $g(t,x) = e^{-4t}\sin(cx)$ satisfies the heat equation $g_t = g_{xx}$.

Answer: $c = \pm 2$, and $c = 0$

Solution: We compute the partial derivatives:

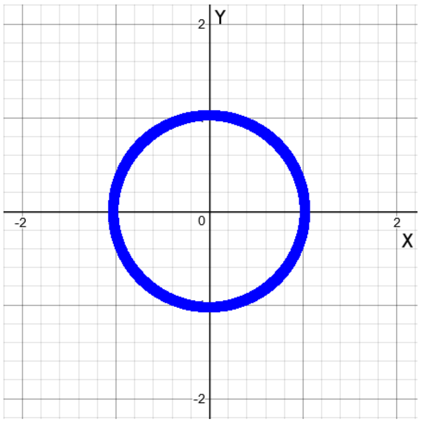
$$g_t(t,x) = -4e^{-4t}\sin(cx),$$
$$g_x(t,x) = ce^{-4t}\cos(cx),$$
$$g_{xx}(t,x) = (-c^2)e^{-4t}\sin(cx).$$

Therefore $g_t = g_{xx}$ leads to $c^2 = 4$, so $c = \pm 2$.
Note also that $c = 0$ is a solution (bonus point for this!)

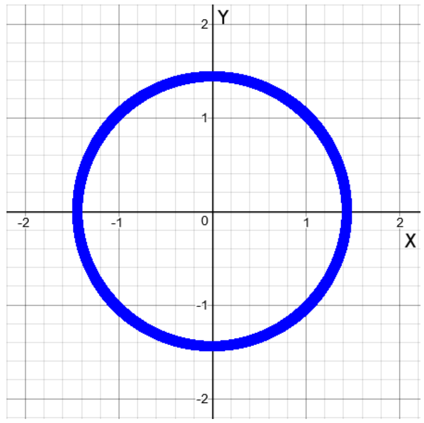
8 marks 3. (a) On the axes provided, draw the level curves of $z = f(x, y)$ at $z = 0, 1$ and 2 for the following functions. *(Note: only those axes will be graded.)*

$f(x, y) = \sqrt{-1 + x^2 + y^2}$

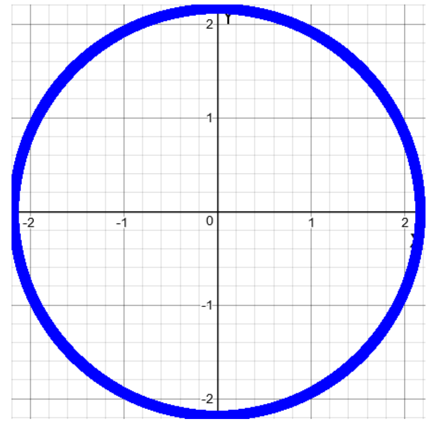
$z = 0$



$z = 1$

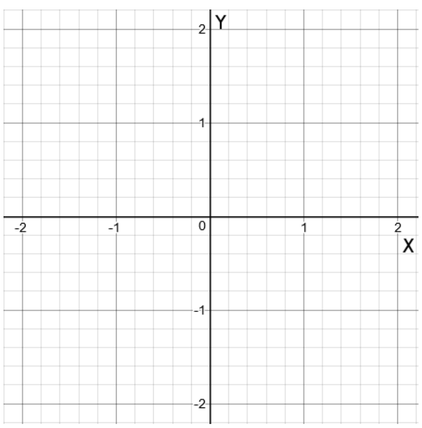


$z = 2$

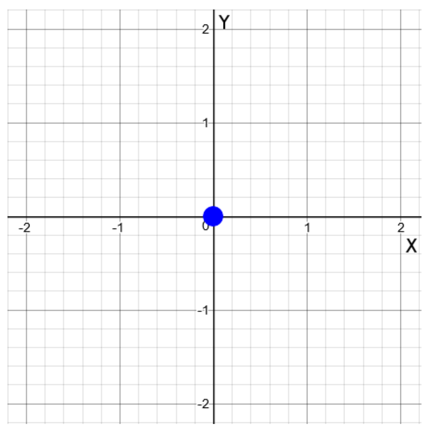


$g(x, y) = \sqrt{1 + x^2 + y^2}$

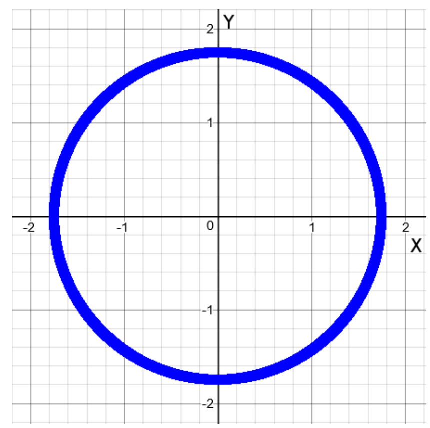
$z = 0$



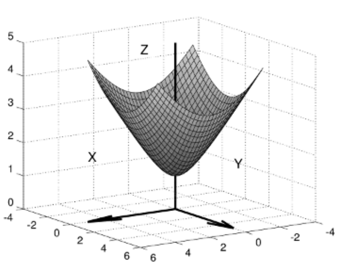
$z = 1$



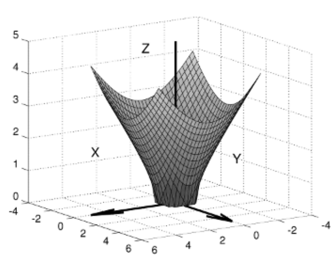
$z = 2$



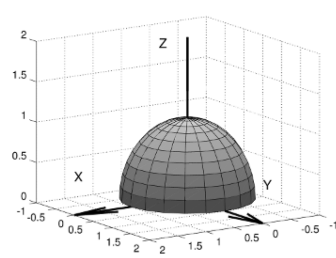
2 marks (b) Of the four graphs pictured below, identify which is a graph of $f(x, y)$ and which is a graph of $g(x, y)$ by writing f or g beneath the appropriate graph.

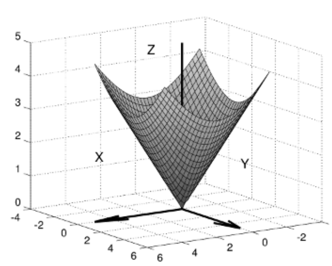


g



f





4. Suppose we are interested in $w(a,b,c) = \dfrac{c^2}{ac-b}$ near $(a_0, b_0, c_0) = (1, 3.9, 4)$.

4 marks (a) Near (a_0, b_0, c_0), the function w is most sensitive to changes in which variable? Briefly justify your answer.

Answer: a, because $|\frac{\partial w}{\partial a}|$ is largest.

Solution:

$$\begin{aligned}
\frac{\partial w}{\partial a} &= -\frac{c^3}{(ac-b)^2} = -\frac{4^3}{(1\cdot 4 - 3.9)^2} = -\frac{64}{0.1^2} = -6400,\\
\frac{\partial w}{\partial b} &= \frac{c^2}{(ac-b)^2} = \frac{16}{0.01} = 1600,\\
\frac{\partial w}{\partial c} &= -\frac{2c(ac-b) - c^2 a}{(ac-b)^2} = 80 - 1600 = -1250.
\end{aligned}$$

Note $\frac{\partial w}{\partial a}$ is largest in magnitude. The total differential of w is thus effected most by changes in a.

2 marks (b) Construct the best linear approximation to w near (a_0, b_0, c_0).

Answer: $160 - 6400(a-1) + 1600(b-3.9) - 1520(c-4)$

Solution: $T(a,b,c) = w(a_0,b_0,c_0) + \frac{\partial w}{\partial a}(a-a_0) + \frac{\partial w}{\partial b}(b-b_0) + \frac{\partial w}{\partial c}(c-c_0)$

2 marks (c) Assuming that each component could vary by up to 0.1 away from (a_0, b_0, c_0), what is the maximum value attained by the linear approximation in part (b)?

Answer: 1112

Solution: Its important to make the largest change in each component by *choosing the signs* appropriately. Specifically, $a = 0.9$, $b = 4.0$, $c = 3.9$, and thus:

$$\begin{aligned}
T(a,b,c) &= 160 - 6400\cdot(0.9-1) + 1600\cdot(4-3.9) - 1520\cdot(3.9-4)\\
&= 160 + 640 + 160 + 152 = 1112.
\end{aligned}$$

2 marks (d) Still assuming that each component is within 0.1 of (a_0, b_0, c_0), how large can the actual value of w become? Briefly justify your answer.

Answer: ∞

Solution: w has a singularity anywhere $ac - b = 0$. Because the numerator of w is positive, this means w can be infinitely large (unbounded). The linear approximation and w disagree so strongly because w is not differentiable at such points. Example of such: $c = 4$, $b = 3.9$ and $a = 3.9/4 = 0.975$.

FYI: motivation for this problem: w is the first component of the inverse of the 2×2 matrix $M = \frac{1}{c}\begin{bmatrix} a & 1 \\ b & c \end{bmatrix}$, which is close to singular for the given (a_0, b_0, c_0).