

Midterm 1 October 12, 2016 Duration: 50 minutes

*This test has **4 questions** on **5 pages**, each worth 10 points, for a total of 40 points.*

- Read all the questions carefully before starting to work.
- Give complete arguments and explanations for all your calculations. Answers without justifications will not be marked, except question #3 where the answer alone is sufficient.
- Continue on the closest blank page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **No aids of any kind are allowed**, including: documents, cheat sheets, electronic devices of any kind (including calculators, phones, etc.)

First Name: Solutions Last Name: _____

Student-No: _____ Section: _____

Signature: _____

Question:	1	2	3	4	Total
Points:	10	10	10	10	40
Score:					

Student Conduct during Examinations

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
 - (i) speaking or communicating with other examination candidates, unless otherwise authorized;
 - (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;
 - (iii) purposely viewing the written papers of other examination candidates;
 - (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
 - (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)—(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

2 marks

1. (a) Consider the three points $A = (0, 1, 1)$, $B = (2, 1, 0)$, $C = (3, -1, 2)$. Find the cosine of the angle between the vectors \overrightarrow{AB} and \overrightarrow{AC} .

Answer: $\sqrt{\frac{5}{14}}$

Solution: The angle θ between the vectors $\overrightarrow{AB} = \langle 2, 0, -1 \rangle$ and $\overrightarrow{AC} = \langle 3, -2, 1 \rangle$ is given by

$$\cos \theta = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{|\overrightarrow{AB}| |\overrightarrow{AC}|} = \frac{5}{\sqrt{5}\sqrt{14}} = \sqrt{\frac{5}{14}}.$$

2 marks

- (b) Continuing from previous part, find a normal vector to the plane containing the points A, B and C .

Answer: $\langle -2, -5, -4 \rangle$

Solution: A normal vector to the desired plane is given by

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & -1 \\ 3 & -2 & 1 \end{vmatrix} = \langle -2, -5, -4 \rangle.$$

3 marks

- (c) Find the equation of the line (in symmetric form) which is at the intersection of the planes $3x - 4y + 2z = -7$ and $3x - 2y + 4z = -5$

Answer: $\frac{x+1}{-2} = \frac{y-1}{-1} = \frac{z}{1}$

Solution: Subtract the two equations to eliminate x , we obtain $y + z = 1$. Eliminating y yields $x + 2z = -1$. Expressing z in terms of x and y , we obtain

$$\frac{x+1}{-2} = \frac{y-1}{-1} = \frac{z}{1}$$

3 marks

- (d) Find the equations of all the planes that are at distance 1 unit from the plane $x + y - z = 1$.

Answer: $x + y - z = 1 \pm \sqrt{3}$

Solution: Two planes separated by a positive distance must be parallel to each other. Therefore the equation of the plane is of the form

$$x + y - z = k.$$

The normal to the planes is $\vec{n} = \langle 1, 1, -1 \rangle$. The points $A = (1, 0, 0)$ and $B = (k, 0, 0)$ are on the planes $x + y - z = 1$ and $x + y - z = k$ respectively. Therefore the distance between the planes is given by the scalar projection of \overrightarrow{AB} on \vec{n} ,

$$\frac{|\overrightarrow{AB} \cdot \vec{n}|}{|\vec{n}|} = \frac{|k - 1|}{\sqrt{3}} = 1.$$

This implies $k = 1 \pm \sqrt{3}$.

4 marks

2. (a) Let $f(x, y) = ye^{2x-y} + 4x \sin(y) + 3x^2$. Compute the partial derivatives f_x and f_y .

$$\text{Answer: } f_x = 2ye^{2x-y} + 4 \sin(y) + 6x, \\ f_y = (1-y)e^{2x-y} + 4x \cos(y).$$

Solution: Partial differentiation simply gives

$$f_x(x, y) = 2ye^{2x-y} + 4 \sin(y) + 6x, \\ f_y(x, y) = e^{2x-y} - ye^{2x-y} + 4x \cos(y).$$

6 marks

- (b) Find all values of the constant c such that $g(t, x) = e^{-4t} \sin(cx)$ satisfies the heat equation $g_t = g_{xx}$.

$$\text{Answer: } c = \pm 2, \text{ and } c = 0$$

Solution: We compute the partial derivatives:

$$g_t(t, x) = -4e^{-4t} \sin(cx), \\ g_x(t, x) = ce^{-4t} \cos(cx), \\ g_{xx}(t, x) = (-c^2)e^{-4t} \sin(cx).$$

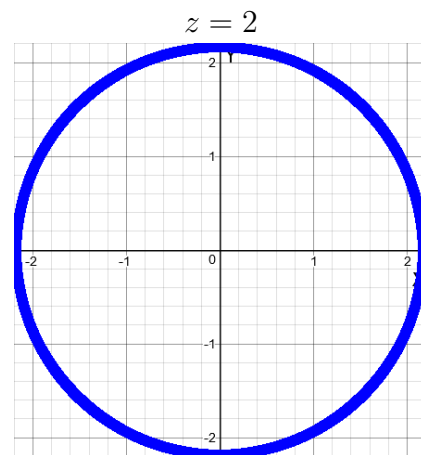
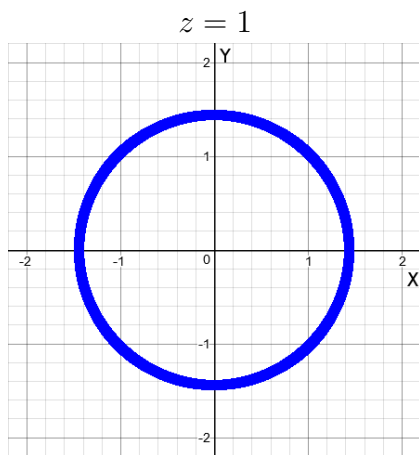
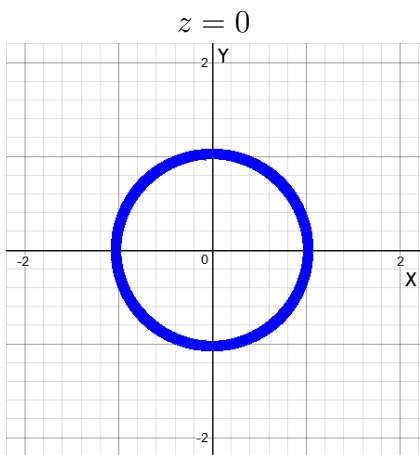
Therefore $g_t = g_{xx}$ leads to $c^2 = 4$, so $c = \pm 2$.

Note also that $c = 0$ is a solution (bonus point for this!)

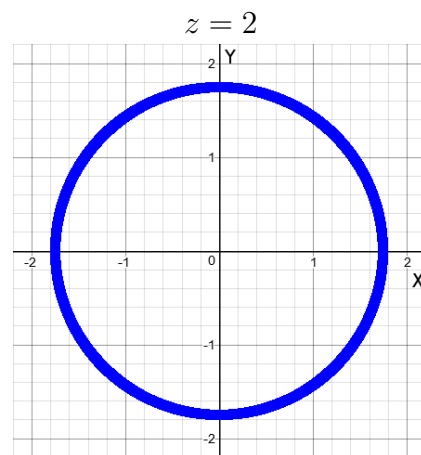
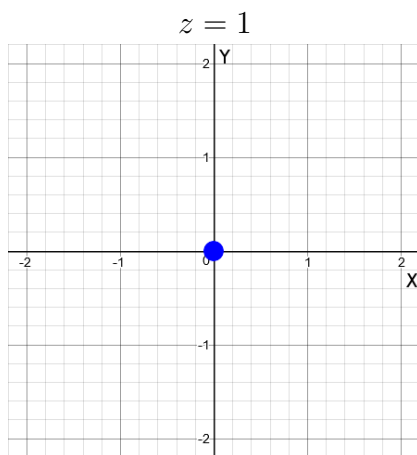
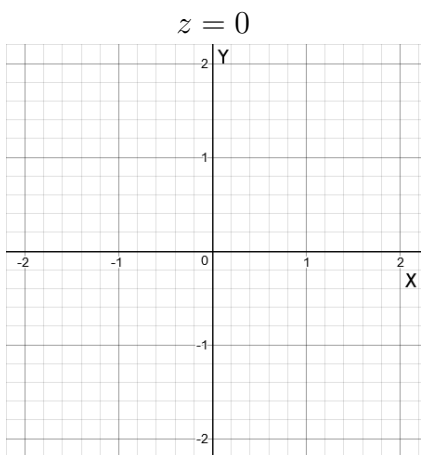
8 marks

3. (a) On the axes provided, draw the level curves of $z = f(x, y)$ at $z = 0, 1$ and 2 for the following functions. (Note: only those axes will be graded.)

$$f(x, y) = \sqrt{-1 + x^2 + y^2}$$

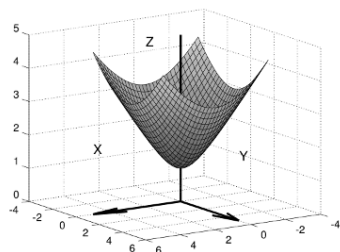


$$g(x, y) = \sqrt{1 + x^2 + y^2}$$

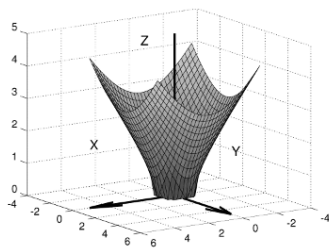


2 marks

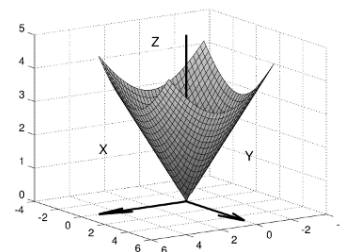
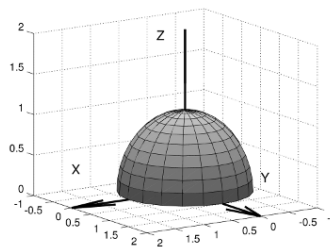
- (b) Of the four graphs pictured below, identify which is a graph of $f(x, y)$ and which is a graph of $g(x, y)$ by writing f or g beneath the appropriate graph.



g



f



4. Suppose we are interested in $w(a, b, c) = \frac{c^2}{ac - b}$ near $(a_0, b_0, c_0) = (1, 3.9, 4)$.

4 marks

- (a) Near (a_0, b_0, c_0) , the function w is most sensitive to changes in which variable? Briefly justify your answer.

Answer: a , because $|\frac{\partial w}{\partial a}|$ is largest.

Solution:

$$\frac{\partial w}{\partial a} = -\frac{c^3}{(ac - b)^2} = -\frac{4^3}{(1 \cdot 4 - 3.9)^2} = -\frac{64}{0.1^2} = -6400,$$

$$\frac{\partial w}{\partial b} = \frac{c^2}{(ac - b)^2} = \frac{16}{0.01} = 1600,$$

$$\frac{\partial w}{\partial c} = -\frac{2c(ac - b) - c^2a}{(ac - b)^2} = 80 - 1600 = -1250.$$

Note $\frac{\partial w}{\partial a}$ is largest in magnitude. The total differential of w is thus effected most by changes in a .

2 marks

- (b) Construct the best linear approximation to w near (a_0, b_0, c_0) .

Answer: $160 - 6400(a - 1) + 1600(b - 3.9) - 1520(c - 4)$

Solution: $T(a, b, c) = w(a_0, b_0, c_0) + \frac{\partial w}{\partial a}(a - a_0) + \frac{\partial w}{\partial b}(b - b_0) + \frac{\partial w}{\partial c}(c - c_0)$

2 marks

- (c) Assuming that each component could vary by up to 0.1 away from (a_0, b_0, c_0) , what is the maximum value attained by the linear approximation in part (b)?

Answer: 1112

Solution: Its important to make the largest change in each component by *choosing the signs* appropriately. Specifically, $a = 0.9$, $b = 4.0$, $c = 3.9$, and thus:

$$\begin{aligned} T(a, b, c) &= 160 - 6400 \cdot (0.9 - 1) + 1600 \cdot (4 - 3.9) - 1520 \cdot (3.9 - 4) \\ &= 160 + 640 + 160 + 152 = 1112. \end{aligned}$$

2 marks

- (d) Still assuming that each component is within 0.1 of (a_0, b_0, c_0) , how large can the actual value of w become? Briefly justify your answer.

Answer: ∞

Solution: w has a singularity anywhere $ac - b = 0$. Because the numerator of w is positive, this means w can be infinitely large (unbounded). The linear approximation and w disagree so strongly because w is not differentiable at such points. Example of such: $c = 4$, $b = 3.9$ and $a = 3.9/4 = 0.975$.

FYI: motivation for this problem: w is the first component of the inverse of the 2×2 matrix $M = \frac{1}{c} \begin{bmatrix} a & 1 \\ b & c \end{bmatrix}$, which is close to singular for the given (a_0, b_0, c_0) .