Final Examination — December 9, 2016 Duration: 2.5 hours

This test has ***8 questions*** *on* ***10 pages****, for a total of 80 points.*

Dr C. Macdonald, Dr M. Murugan, Dr R. Balka, Dr N. Bade

- Read all the questions carefully before starting to work. Unless otherwise indicated, give complete arguments and explanations for all your calculations as answers without justification will not be marked.
- Continue on the back of the *previous page* if you run out of space, *with clear indication on the original page* that your solution is continued elsewhere.
- This is a closed-book examination. **No aids of any kind are allowed**, including: documents, cheat sheets, electronic devices of any kind (including calculators, phones, etc.)

First Name: Solutions Last Name: ____________

Student-No: ____________ Section: ____________

Signature: ____________

Question:	1	2	3	4	5	6	7	8	Total
Points:	14	15	10	6	7	14	8	6	80
Score:									

5 marks 1. (a) Suppose $z = f(x, y)$ describes the surface of a mountain. The values of some derivatives are known at certain points:

point	f_x	f_y	f_{xx}	f_{yy}	f_{xy}
A	0	1	4	-3	-2
B	0	0	4	3	-4
C	10	2	-5	3	2
E	0	0	-3	-3	2
F	0	0	5	5	5
G	0	0	3	4	3

For each situation, identify a suitable point **and *briefly* explain:** e.g., "X, because it is a local minimum." **Justify your answers with calculations.**

i. Mika wants to hike to point with view in every direction (where the mountain does not obscure her view). Where should she go?

Answer: E, local max

ii. Xiaofei is a biologist who wants to re-introduce a species of frog into a wet environment. Where is water mostly likely to form a pool?

Answer: G, local min

iii. Donnie wants to have picnic at a place where the ground is flat and dry. He thinks a peak would be too windy. Where should he go?

Answer: B, saddle

iv. Akshat also wants to have a picnic where the ground is flat, but he likes surprises. Where should he go?

Answer: F, tests are inconclusive

Solution: Calculate $D = f_{xx}f_{yy} - f_{xy}^2$ for each case where $f_x = 0 = f_y$.
A: not a CP.
B: $D = 4 \cdot 3 - 4^2 = -4 < 0$: saddle.
C: not a CP.
E: $D = 9 - 4 > 0$ and $f_{xx} < 0$: local max.
F: $D = 25 - 25 = 0$: test inconclusive, so find out when you get there.
G: $D = 12 - 9 = 3$ and $f_{xx} > 0$: local min.

2 marks

(b) Suppose $f(x, y)$ is some function, unrelated to previous problem. At a point (x, y), what is the maximal value of the directional derivative $D_{\vec{u}}f$?

Answer: $|\vec{\nabla} f(x, y)|$

Solution: $D_{\vec{u}}f = \vec{\nabla} f \cdot \vec{u}$ so we can maximize this by choosing $\vec{u}$ to be parallel to $\vec{\nabla} f$. Also $\vec{u}$ should be a unit vector so $\vec{u} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|}$. Thus:

$$D_{\frac{\vec{\nabla} f}{|\vec{\nabla} f|}} f = \vec{\nabla} f \cdot \frac{\vec{\nabla} f}{|\vec{\nabla} f|} = \frac{|\vec{\nabla} f|^2}{|\vec{\nabla} f|} = |\vec{\nabla} f|.$$

Note $|\vec{\nabla} f|$ can have different values at different points: that's what the rest of this problem is about.

7 marks

(c) Suppose $f(x, y)$ is some function. Define $g(x, y) = |\vec{\nabla} f|^2 = (f_x(x, y))^2 + (f_y(x, y))^2$.

i. What is $\vec{\nabla} g$?

Solution: Use the chain rule:
$g_x = \frac{\partial (f_x)^2}{\partial x} + \frac{\partial (f_y)^2}{\partial x} = 2f_x \frac{\partial f_x}{\partial x} + 2f_y \frac{\partial f_y}{\partial x} = 2f_x f_{xx} + 2f_y f_{xy}$;
$g_y = 2f_x \frac{\partial f_x}{\partial y} + 2f_y \frac{\partial f_y}{\partial y} = 2f_x f_{xy} + 2f_y f_{yy}$.

ii. Consider a mountain where the derivatives are $f_x(x, y) = \exp(-x^2)$ and $f_y(x, y) = \exp(-y^4 + 2y^2)$. Find the critical points of $g(x, y)$ (no need to classify).

Answer: $(x, y) = (0, 0), (0, 1)$ and $(0, -1)$

Solution: Use the previous result. Here $f_{xy} = f_{yx} = 0$ and $f_{xx} = -2xe^{-x^2}$ and $f_{yy} = (-4y^3 + 4y)e^{-y^4+2y^2}$.
So $g_x = 2e^{-x^2}(-2xe^{-x^2}) = 0 \implies x = 0$.
And $g_y = 2e^{-y^4+2y^2}(-4y^3 + 4y)e^{-y^4+2y^2} \implies y(1 - y^2) = 0 \implies y = 0, 1, -1$.

iii. What is the geometric interpretation (in terms of f) of the point (x, y) where $g(x, y)$ is maximal?

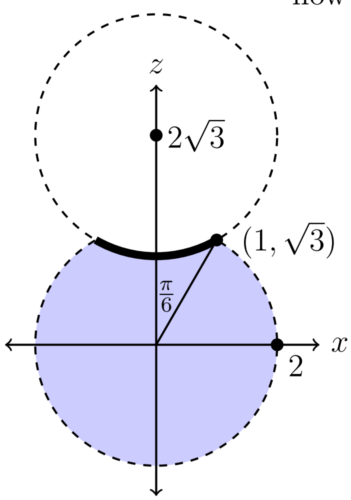
Solution: Steepest place the surface $z = f(x, y)$.
$g(x, y)$ measures the square of $|\vec{\nabla} f(x, y)|$ which is the maximum directional derivative at (x, y). Thus maximizing g means finding the point where the surface $z = f(x, y)$ is steepest: the steepest point on the mountain.

2 marks 2. (a) To find the surface area of the surface $z = f(x, y)$ above the region D, we integrate $\iint_D F(x, y)\,\mathrm{d}A$. What is $F(x, y)$?

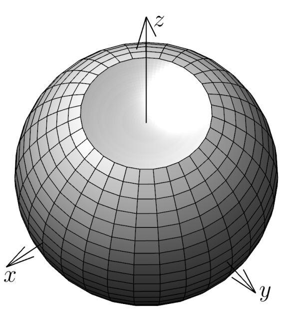
Answer: $\sqrt{1 + f_x^2 + f_y^2}$.

(b) Consider a "Death Star", a ball of radius 2 centred at the origin with another ball of radius 2 centred at $(0, 0, 2\sqrt{3})$ cut out of it. The diagram shows the slice where $y = 0$.

4 marks i. The Rebels want to paint *part* of the surface of Death Star hot pink; specifically, the concave part (indicated with a thick line in the diagram). To help them determine how much paint is needed, carefully fill in the missing parts of this integral:



$$\text{surface area} = \int_{\boxed{}}^{\boxed{}} \int_{\boxed{}}^{\boxed{}} \boxed{}\, \mathrm{d}r\, \mathrm{d}\theta.$$



Solution: First limits are 0 to 2π. Second limits are 0 to 1.
Equation for surface: look at the top (cutting) sphere: $x^2 + y^2 + (z - 2\sqrt{3})^2 = 4$, solve for bottom bit: $z = f(x, y) := 2\sqrt{3} - \sqrt{4 - x^2 - y^2}$.
Now, need the partials $z_x = \frac{x}{\sqrt{4-x^2-y^2}}$ and $z_y = \frac{x}{\sqrt{4-x^2-y^2}}$.
And we have:

$$\int_0^{2\pi} \int_0^1 \sqrt{1 + \frac{x^2}{4-x^2-y^2} + \frac{y^2}{4-x^2-y^2}}\, r\, \mathrm{d}r\, \mathrm{d}\theta = \int_0^{2\pi} \int_0^1 \sqrt{\frac{4}{4 - r^2}}\, r\, \mathrm{d}r\, \mathrm{d}\theta.$$

(Don't forget the r for "$r\,\mathrm{d}r\,\mathrm{d}\theta$".)
Alternatively, you can use $z = f(x, y) = \sqrt{4 - x^2 - y^2}$ as this will have the same surface area.

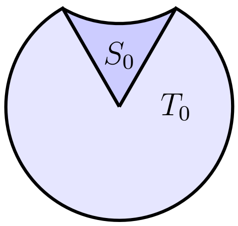
1 mark ii. What is the total surface area of the Death Star?

Answer: $4\pi 2^2 = 16\pi$

Solution: The concave bit has the same surface area as that part of the original sphere. So answer is just the surface area of a sphere of radius 2 (no calculation required).

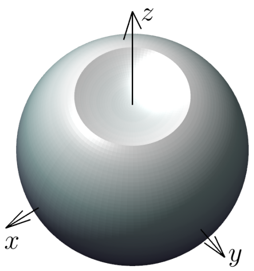
8 marks

(c) Suppose the solid interior of the Death Star has density T_0, except in a cone connecting the origin to the concave region where the density is S_0. Complete this expression for the total mass M:



$$M = \int_{\square}^{\square}\int_{\square}^{\square}\int_{\square}^{\square} T_0 \,\square\, \mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$$

$$+ \int_{\square}^{\square}\int_{\square}^{\square}\int_{\square}^{\square} S_0 \,\square\, \mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$$



Solution:

$$M = \int_0^{2\pi}\int_{\frac{\pi}{6}}^{\pi}\int_0^{2} T_0\,\rho^2 \sin\phi\ \mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta + \int_0^{2\pi}\int_0^{1}\int_{\sqrt{3}r}^{2\sqrt{3}-\sqrt{4-r^2}} S_0\, r\ \mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$$

2 marks 3. (a) Give the formula for the linear approximate of $h(x,y)$ near the point (x_0, y_0).

$h(x,y) \approx$

Solution: $h(x,y) \approx h(x_0,y_0) + h_x(x_0,y_0)(x-x_0) + h_y(x_0,y_0)(y-y_0)$

4 marks (b) The equation $z^5 - 2z^3 - x^2y - 7y = -33$ determines a function $z = h(x,y)$ implicitly where $h(3,2) = 1$. Find the linear approximation of h near $(x_0, y_0) = (3,2)$.

Answer: $h(x,y) \approx 1 - 12(x-3) - 16(y-2)$.

Solution: We differentiate implicitly to obtain

$$5z^4\frac{\partial z}{\partial x} - 6z^2\frac{\partial z}{\partial x} - 2xy = 0, \quad 5z^4\frac{\partial z}{\partial y} - 6z^2\frac{\partial z}{\partial y} - x^2 - 7 = 0$$

which gives

$$\frac{\partial z}{\partial x}(x,y) = \frac{2xy}{5z^4 - 6z^2}, \quad \frac{\partial z}{\partial x}(x,y) = \frac{x^2+7}{5z^4 - 6z^2}.$$

At $(x,y) = (3,2)$, where $z = 1$, we have $h_x(3,2) = -12, h_y(3,2) = -16$. Therefore the linear approximation is

$$h(x,y) \approx h(3,2) + h_x(3,2)(x-3) + h_y(3,2)(y-2) = 1 - 12(x-3) - 16(y-2).$$

4 marks (c) The kinetic energy K of an object rotating with angular velocity $\omega = 3$ radian per second and moment of inertia $I = 10\,\text{Kg m}^2$ is given by $\text{K} = \frac{1}{2}I\omega^2$. However, the moment of inertia has an error of up to $\pm 0.1\,\text{Kg m}^2$ and the angular velocity has an error of up to ± 0.02 radian per second. Use differentials to estimate the maximal possible error in the computed kinetic energy K.

Answer: 1.05 Joules

Solution: We compute the differential as

$$dK = \frac{1}{2}\omega^2\, dI + I\omega\, d\omega.$$

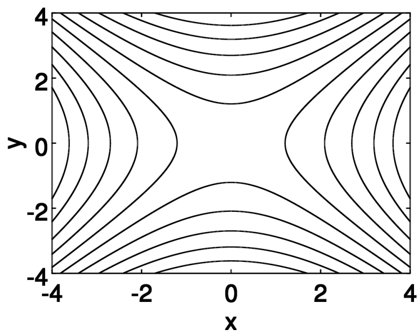
The maximum possible error in the kinetic energy is estimated as

$$dK = \frac{1}{2} \times 3^2 \times 0.1 + 10 \times 3 \ \times 0.02 = 1.05\,\text{Kgm}^2/\text{s}^2 = 1.05\,\text{Joule}\,.$$

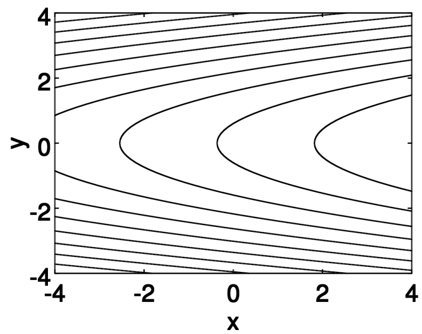
$\boxed{\text{6 marks}}$ 4. Match the following functions with their contour plot.

C $f(x,y) = xe^{-x^2-y^2}$, **J** $f(x,y) = \sin(x) - \cos(y)$, **E** $f(x,y) = 3x^2 - y$,

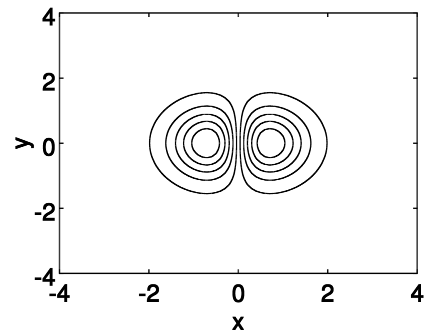
D $f(x,y) = \sqrt{12 - 4x^2 - y^2}$, **F** $f(x,y) = xy$. **A** $f(x,y) = x^2 - y^2$.



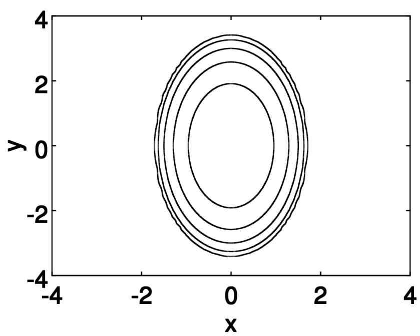
(A)



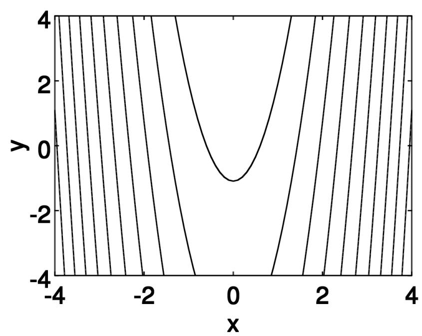
(B)



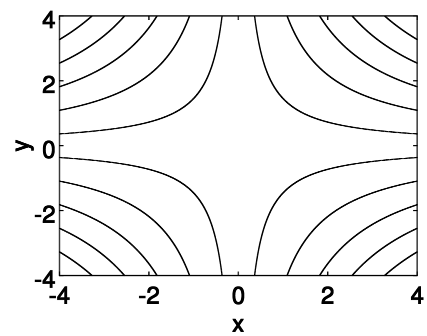
(C)



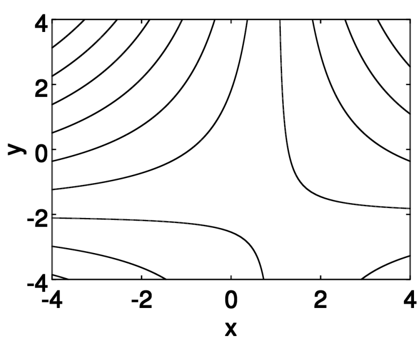
(D)



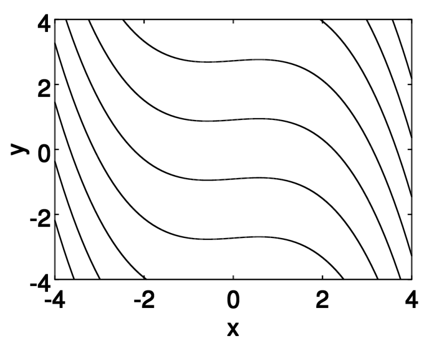
(E)



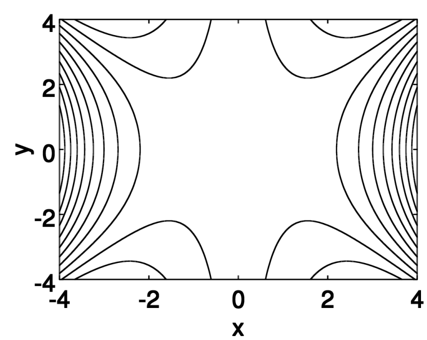
(F)



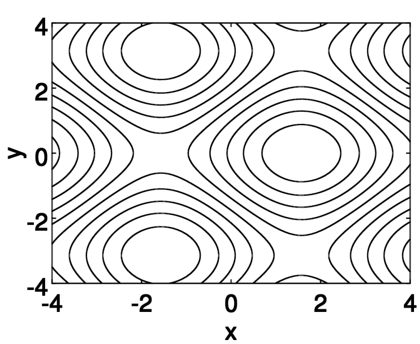
(G)



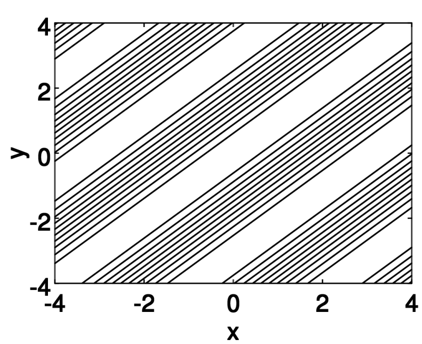
(H)



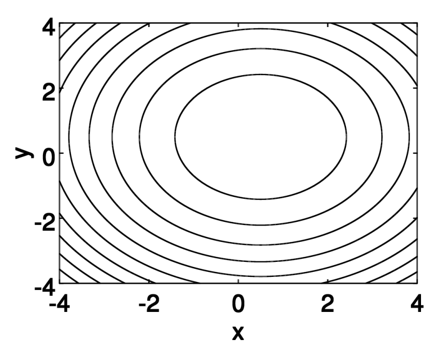
(I)



(J)

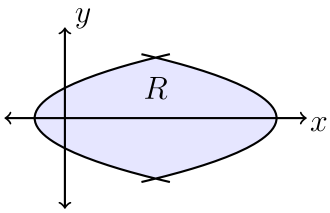


(K)



(L)

4 marks 5. (a) Let R be the region bounded on the left by $x = y^2 - 1$ and on the right by $x = -y^2 + 7$. Evaluate the integral $\iint_R 3 \, \mathrm{d}A$.



Answer: 64

Solution: Endpoints: $y^2 - 1 = -y^2 + 7$ gives $2y^2 = 8$ or $y = \pm 2$ as the intersection points. Then

$$\begin{aligned}\int_{-2}^{2}\int_{y^2-1}^{-y^2+7} 3\,dx\,dy &= 3\int_{-2}^{2}(-y^2+7)-(y^2-1)\,dy\\ &= 3\int_{-2}^{2} -2y^2+8\,dy\\ &= 3\cdot\left(-\frac{2}{3}y^3+8y\right)\Big|_{-2}^{2}\,dy\\ &= 64\end{aligned}$$

1 mark (b) What is the area of R?

Answer: 64/3

2 marks (c) Evaluate the integral

$$\int_{-2}^{2}\int_{y^2-1}^{-y^2+7} y\cos\left(y^6\right)\cos(x)\,\mathrm{d}x\,\mathrm{d}y \; =$$

Answer: 0, by symmetry

Solution: Note R is symmetric with respect to the x-axis and $f(x, y)$ is odd in y.

8 marks 6. (a) Let

$$f(x,y,z) = e^{\left((x^2+y^2+z^2)^{3/2}\right)}.$$

Rewrite

$$\int_0^3 \int_{-\sqrt{(9-y^2)}}^{\sqrt{(9-y^2)}} \int_{\sqrt{(x^2+y^2)}}^{\sqrt{(18-x^2-y^2)}} f(x,y,z)\,dz\,dx\,dy$$

using spherical coordinates. It *is not necessary* to evaluate.

Solution: We first express the region

$$E = \left\{(x,y,z) : \sqrt{(x^2+y^2)} \le z \le \sqrt{(18-x^2-y^2)}, -\sqrt{(9-y^2)} \le x \le \sqrt{(9-y^2)}, 0 \le y \le 3\right\}.$$

This region looks like "half of an ice cream cone" and can be expressed in spherical coordinates as

$$E = \left\{(\rho,\theta,\phi) : 0 \le \rho \le 3\sqrt{2}, 0 \le \theta \le \pi, 0 \le \phi \le \pi/4\right\}.$$

Notice the that function f in spherical coordinates is e^{ρ^3}. Therefore

$$\iiint_E f(x,y,z)\,dV = \int_0^{\pi/4} \int_0^{\pi} \int_0^{3\sqrt{2}} e^{\rho^3} \rho^2 \sin(\phi)\,d\rho\,d\theta\,d\phi$$

6 marks (b) Let

$$\int_0^4 \int_{-2}^{2-z} \int_{-\sqrt{4-y^2}}^{0} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,.$$

Change the order of integration to be $\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$.

Solution: The region of integration is the volume inside the cylindar of radius 2, under the plane $z = 2 - y$, above the plane $z = 0$, and with $x \le 0$. The integral in the new order is given by

$$\int_{-2}^{0} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{2-y} f(x,y,z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x\,.$$

Alternatively, this can be done by only manipulating double integrals to permute the order pairwise:

- Change order of outer two: $\mathrm{d}y\,\mathrm{d}z$ to $\mathrm{d}z\,\mathrm{d}y$.
- Now inner two integrals are over a rectangular region so we can invoke 2D Fubini's Theorem. Thus we have $\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$.
- Change order of outer two: $\mathrm{d}x\,\mathrm{d}y$ to $\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$.

8 marks

7. Calculate the volume of the region E enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 9$.

Solution: Using cylindrical coordinates E can be written as

$$E = \{(r, \theta, z) : 0 \le r \le 3,\ r^2 \le z \le 9\},$$

so the volume is

$$\iiint_E 1\,\mathrm{d}V = \int_0^3 \int_0^{2\pi} \int_{r^2}^9 r\,\mathrm{d}z\,\mathrm{d}\theta\,\mathrm{d}r = \int_0^3 2\pi r(9 - r^2)\,\mathrm{d}r = (\pi/2)(18r^2 - r^4)\Big|_{r=0}^{r=3} = \frac{81\pi}{2}.$$

8. Consider the planes $z = 4x - 2y - 3$ and $z = 8x - 4y - 12$.

2 marks

(a) Determine the cosine of the acute angles between the two planes.

Solution:
The vectors $\vec{n}_1 = \langle 4, -2, -1\rangle$ and $\vec{n}_2 = \langle 8, -4, -1\rangle$ are normals to the planes. The acute angle θ between the planes is either the angle between the normal vectors or its complementary angle, so we have

$$\cos\theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1||\vec{n}_2|} = \frac{41}{9\sqrt{21}}.$$

4 marks

(b) Determine the equation of the line of intersection of the two planes. Give your answer in parametric form. At what point (x, y, z) does that line intersect with the plane $x = 1$?

Answer: $(1, -5/2, 6)$

Solution: First we find a point on the line by solving $4x - 2y - 3 = 8x - 4y - 12$, that is, $4x - 2y = 9$. Thus $x = 3$ and $y = 3/2$ works, for which $z = 6$. This gives a point

$$P = (3, 3/2, 6).$$

The cross product

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -2 & -1 \\ 8 & -4 & -1 \end{vmatrix} = \langle -2, -4, 0\rangle$$

is parallel to the line of intersection. Thus the parametric form is

$$\langle x, y, z\rangle = \langle 3, 3/2, 6\rangle + t\langle -2, -4, 0\rangle = \langle 3 - 2t, 3/2 - 4t, 6\rangle.$$

Intersection bit: set $3 - 2t = 1$, this gives $-2t = -2$ or $t = 1$. Sub into the line and we get the point $(1, -5/2, 6)$.