Final Examination — December 6, 2014 Duration: 2.5 hours

*This test has 8 **questions** on 17 **pages**, for a total of 80 points.*

- Read all the questions carefully before starting to work.
- Give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- This is a closed-book examination. **No aids of any kind are allowed**, including: documents, cheat sheets, electronic devices of any kind (including calculators, phones, etc.)

First Name: ______________________ Last Name: ______________________

Student-No: ______________________ Section: ______________________

Signature: ____

Question:	1	2	3	4	5	6	7	8	Total
Points:	10	10	11	9	8	8	10	14	80
Score:									

3 marks 1. (a) Find the equation of the tangent plane to the surface $x^2 + y^2 + z^2 = 9$ at the point $(2, 2, 1)$ on that surface.

Answer: $2x + 2y + z = 9$

Solution: The surface is given by $F(x, y, z) = x^2 + y^2 + z^2 - 9 = 0$. Since $\nabla F = \langle 2x, 2y, 2z \rangle$, the normal to the plane at the given point is $\nabla F(2, 2, 1) = \langle 4, 4, 2 \rangle$ and hence the tangent plane is $4x + 4y + 2z = 4 \cdot 2 + 4 \cdot 2 + 2 \cdot 1 = 18$, or $2x + 2y + z = 9$.

2 marks (b) Find the equation of the tangent plane to the surface $x^2 + y^2 - 8z^2 = 0$ at the point $(2, 2, 1)$ on that surface.

Answer: $x + y - 4z = 0$

Solution: The new surface is $G(x, y, z) = x^2 + y^2 - 8z^2 = 0$. Since $\nabla G = \langle 2x, 2y, -16z \rangle$, the normal to the plane at the given point is $\nabla F(2, 2, 1) = \langle 4, 4, -16 \rangle$ and hence the tangent plane is $4x + 4y - 16z = 4 \cdot 2 + 4 \cdot 2 - 16 \cdot 1 = 0$, or $x + y - 4z = 0$.

3 marks (c) The planes of parts (a) and (b) intersect in a line which passes through the point $(2,2,1)$. Write an equation for that line of intersection, in parametric form.

Answer: $x = 2+t$, $y = 2-t$, $z = 1$

Solution: The line of intersection of the two planes obeys $2x + 2y + z = 9$ and $x + y - 4z = 0$, from which we find $2x + 2y = 9 - z$ and $2x + 2y = 8z$, so $9 - z = 8z$ and $z = 1$. This gives the line $z = 1$, $2x + 2y = 8$. We set $x = 2 + t$ and find $y = 2 - t$.

2 marks (d) Find the points on the line of part (c) which are distance 1 from $(2,2,1)$.

Answer: $(2 + \frac{1}{\sqrt{2}}, 2 - \frac{1}{\sqrt{2}}, 1)$, $(2 - \frac{1}{\sqrt{2}}, 2 + \frac{1}{\sqrt{2}}, 1)$

Solution: The line is $x = 2+t$, $y = 2-t$, $z = 1$, so the direction vector of the line is $\langle 1, -1, 0\rangle$, and the unit vectors in this direction are $\pm\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\rangle$. Therefore the points that are distance 1 away are given by $(2,2,1) \pm (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0)$.

5 marks 2. (a) If two resistors with resistances R_1, R_2 are connected in parallel, then the total resistance R is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

The resistances are measured in ohms as $R_1 = 30\Omega$, $R_2 = 40\Omega$, with a possible error of 0.5% in each case. Use differentials to estimate the maximum error in the calculated value of R.

Answer: $\frac{3}{35}\Omega$

Solution: Differentiation of both sides with respect to R_1 gives $-R^{-2}\frac{\partial R}{\partial R_1} = -R_1^{-2}$, so $\frac{\partial R}{\partial R_1} = (R/R_1)^2$. Similarly, $\frac{\partial R}{\partial R_2} = (R/R_2)^2$. For the given values, $R = \frac{1200}{70}$. With the errors $.005R_i$, we obtain

$$\begin{aligned} dR &= (R/R_1)^2(.005R_1) + (R/R_2)^2(.005R_2) \\ &= .005R^2\left((1/R_1)^2R_1 + (1/R_2)^2R_2\right) \\ &= .005R^2R^{-1} = .005R = \frac{3}{35}. \end{aligned}$$

5 marks

(b) The length x of a side of a triangle is increasing at a rate of 3 cm/s, the length y of another side is decreasing at a rate of 2 cm/s, and the included angle θ is increasing at a rate of 0.05 radian/s. How fast is the area of the triangle changing when $x = 40$ cm, $y = 50$ cm, $\theta = \pi/6$? (Recall that the area of a triangle is one half base times height.)

Answer: $\frac{1}{2}(35 + 50\sqrt{3})$ cm/s

Solution: $A = \frac{1}{2}xy\sin\theta$, so

$$\begin{aligned}\frac{dA}{dt} &= \frac{\partial A}{\partial x}\frac{dx}{dt} + \frac{\partial A}{\partial y}\frac{dy}{dt} + \frac{\partial A}{\partial \theta}\frac{d\theta}{dt}\\ &= \frac{1}{2}y\sin\theta \times 3 + \frac{1}{2}x\sin\theta \times (-2) + \frac{1}{2}xy\cos\theta \times .05\end{aligned}$$

With the given dimensions,

$$\begin{aligned}\frac{dA}{dt} &= \frac{1}{2}50\frac{1}{2} \times 3 + \frac{1}{2}40\frac{1}{2} \times (-2) + \frac{1}{2}(40)(50)\frac{\sqrt{3}}{2} \times .05\\ &= \frac{1}{2}(35 + 50\sqrt{3}).\end{aligned}$$

3. Let $F(x, y) = (x^2 - 1)(x^2 + y^2)$, for $(x, y) \in \mathbb{R}^2$.

4 marks (a) Find all critical points of F.

Answer: $(0,0)$, $(\pm\frac{1}{\sqrt{2}}, 0)$.

Solution: The partial derivatives are

$$F_x = 2x(2x^2 - 1 + y^2), \qquad F_y = 2y(x^2 - 1).$$

$F_x = 0$ iff $x = 0$ or $2x^2 - 1 + y^2 = 0$, and $F_y = 0$ iff $y = 0$ or $x^2 - 1 = 0$.
If $x^2 - 1 = 0$ in $F_y = 0$, then one cannot have $x = 0$ in $F_x = 0$, so $2x^2 - 1 + y^2 = 0$ in $F_x = 0$. This implies that $1 + y^2 = 0$, and there is no solution.
On the other hand, if $y = 0$ in $F_y = 0$, then either $x = 0$ or $2x^2 - 1 = 0$ in $F_x = 0$.
This gives the 3 critical points: $P_1 = (0,0)$, $P_2 = (\frac{1}{\sqrt{2}}, 0)$, $P_3 = (-\frac{1}{\sqrt{2}}, 0)$.

3 marks (b) Classify each critical point from part (a) as local maximum, local minimum, or saddle point.

Answer: local maximum $(0,0)$, saddle points $(\pm\frac{1}{\sqrt{2}}, 0)$.

Solution: The second partial derivatives are:

$$F_{xx} = 2(2x^2 - 1 + y^2) + 8x^2, \qquad F_{xy} = 4xy, \qquad F_{yy} = 2(x^2 - 1).$$

Thus, $F_{xy}(P_i) = 0$ for $i = 1, 2, 3$. Also, $F_{xx}(P_1) = -2$, $F_{yy}(P_1) = -2$,
$F_{xx}(P_2) = F_{xx}(P_3) = 4$, $F_{yy}(P_2) = F_{yy}(P_3) = -1$.
Let $D = F_{xx}F_{yy} - F_{xy}^2$. Then $D(P_1) > 0$ and $D(P_i) < 0$ for $i = 1, 2$.
We conclude that P_1 is a local maximum, and P_2, P_3 are saddle points.

3 marks

(c) Find the maximum and minimum of F on the circle $\{(x,y) \mid x^2+y^2=4\}$.
(Hint: do *not* use Lagrange multipliers).

Answer: max: 12, min: -4.

Solution: We insert $x^2+y^2=4$ in the expression giving F, so that on the boundary of D we have $F(x,y) = 4(x^2-1)$. On the boundary, x belongs to $[-2,2]$, so the maximum of F on the boundary is $4 \times 3 = 12$, and the minimum is -4.

1 mark

(d) Find the absolute maximum and minimum of F on $D = \{(x,y) \mid x^2+y^2 \leq 4\}$. Carefully justify each step in your answer.

Answer: max: 12, min: -4

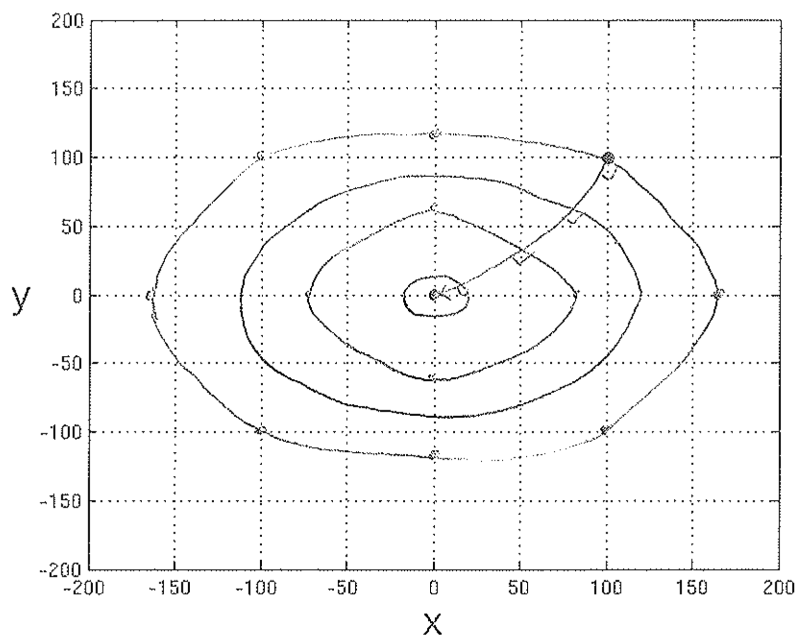
Solution:
The max/min of F is either attained at a local max/local min in the interior of D, or at the boundary. In the interior, there is a local max P_1, and $F(P_1) = 0$. There is no local min in the interior of D. At the boundary, we have computed that the max is 12 and the min is -4. Therefore, the absolute max of F in D is 12, and the absolute min is -4.

4. When a shark detects blood in the water, it will swim in the direction in which the concentration of blood increases most rapidly. Suppose that the concentration of blood (in parts per million) at a point (x, y) on the surface of seawater is given approximately by

$$C(x, y) = e^{-(x^2+2y^2)/10^4},$$

where x and y are measured in metres in a rectangular coordinate system with the blood source at the origin. The concentration is assumed not to change during the time period under consideration.

4 marks (a) Identify the level curves of the concentration function and sketch at least four members of this family of curves. Your sketch should be accurate and clearly labelled with details carefully displayed. Accurately sketch the path that a shark would follow towards the source if it starts at the point (100, 100), and explain why this is the shark's path.



Solution: The level curves are ellipses with aspect ratio (vertical/horizontal) equal to $2^{-1/2} \approx 0.7$. The sketch shows four such ellipses, including the one through (100, 100). The concentration of blood increases most rapidly in the direction of the gradient, which is orthogonal to the level curves and pointing inwards. The shark moves in the direction of the gradient, and to keep perpendicular to each level curve, will follow the curve sketched.

(100,100) is on level curve $C(x,y) = e^{-3}$
with intercepts $x = \sqrt{3} \times 100 \approx 170$ & $y = \sqrt{\frac{3}{2}} \times 100 \approx 170 \times .7 = 120$

other level curves are similar ellipses, with concentration increasing towards the centre.

3 marks

(b) What is the slope of the tangent to the shark's path as it passes through a point (x, y)?

Answer:

Solution: The shark moves in the direction of the gradient $\nabla C = -\frac{2C}{10^4}\langle x, 2y\rangle$. The slope is therefore $\frac{2y}{x}$.

2 marks

(c) Suppose the shark starts at the point $(100, 100)$. Its path after it has swum to the origin is given by $y = f(x)$, for some function f. Determine this function f by first setting $\frac{dy}{dx}$ equal to the slope found in part (b), and then solving this differential equation.

Answer: $y = \frac{1}{100}x^2$

Solution: The equation $\frac{dy}{dx} = \frac{2y}{x}$ can be written as $\frac{1}{2y}dy = \frac{1}{x}dx$, and integration then gives $\frac{1}{2}\ln y = \ln x + C$, or $y = cx^2$ for some c. We choose $c = \frac{1}{100}$ for the curve to contain $(100, 100)$, so the shark follows the curve $y = \frac{1}{100}x^2$.

3 marks 5. (a) Let D be the bounded planar region enclosed by the hyperbola $x^2 - y^2 = 1$ and the line $y = -2x + 2$. Give a careful sketch of the region D, with all relevant details labelled.

Solution:

3 marks (b) Write the integral $\int\int_D y dA$ as an iterated integral. Do not evaluate the integral.

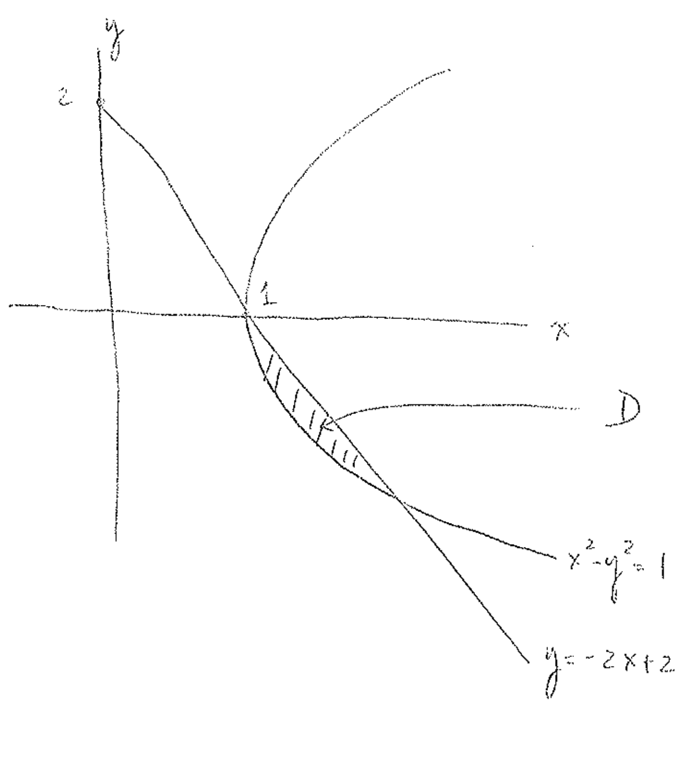
Answer:

$$\int_{-\frac{4}{3}}^{0} \int_{\sqrt{1+y^2}}^{1-\frac{1}{2}y} y dx dy$$

Solution: First we compute the points of intersection between the curves $x^2 - y^2 = 1$ and $y = -2x + 2$. We substitute $y = -2x + 2$ in the first equation to get $3x^2 - 8x + 5 = 0$. Solutions are given by $x = \frac{8 \pm 2}{6}$, i.e. $x = \frac{5}{3}$ and $x = 1$. The points of intersection are $(1, 0)$ and $(\frac{5}{3}, -\frac{4}{3})$. The region D is of Type II (or Type I), and we can write $D = \{(x, y) | -\frac{4}{3} \leq y \leq 0, \sqrt{1+y^2} \leq x \leq 1 - \frac{1}{2}y\}$. Thus we have:

$$\int\int_D y dA = \int_{-\frac{4}{3}}^{0} \int_{\sqrt{1+y^2}}^{1-\frac{1}{2}y} y dx dy.$$

(a)



2 marks (c) Evaluate $I = \int\int_D y dA$.

Answer: $-\frac{2}{27}$

Solution:

$$I = \int_{-\frac{4}{3}}^{0} \int_{\sqrt{1+y^2}}^{1-\frac{1}{2}y} y dx dy = \int_{-\frac{4}{3}}^{0} yx\Big|_{x=\sqrt{1+y^2}}^{x=1-\frac{1}{2}y} dy = \int_{-\frac{4}{3}}^{0} (y - \frac{1}{2}y^2) dy - \int_{-\frac{4}{3}}^{0} y\sqrt{1+y^2} dy$$

$$= [\frac{1}{2}y^2 - \frac{1}{6}y^3]_{-\frac{4}{3}}^{0} - [\frac{2}{6}(1+y^2)^{\frac{3}{2}}]_{-\frac{4}{3}}^{0} = -\frac{104}{81} + \frac{98}{81} = -\frac{2}{27}.$$

3 marks 6. (a) The average of a function $f(x,y)$ on a planar region D is given by the formula $f_{\text{ave}} = \frac{1}{\text{Area}(D)} \int\int_D f dA$. Find the average distance between a point lying inside a circle of radius 1 and the centre of the circle.

Answer: $\frac{2}{3}$

Solution: We use polar coordinates and write $D = \{(r,\theta) \mid 0 \leq \theta \leq 2\pi, 0 \leq r \leq 1\}$. The area of D is π. The average distance of a point lying inside the circle is given by:

$$\frac{1}{\pi}\int_0^{2\pi}\int_0^1 r \cdot r dr d\theta = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{3}r^3\Big|_0^1 d\theta = \frac{2}{3}.$$

3 marks (b) Consider the planar region $D = \{(x,y) \mid x \geq 0, (x-1)^2 + y^2 \geq 1, x^2 + y^2 \leq 4\}$. Sketch the region D and describe it in polar coordinates.

Solution: Let S_1 be the circle of radius 2 and centre $(0,0)$, and let S_2 be the circle of radius 1 and centre $(1,0)$. In polar coordinates S_1 has equation $r = 2$, and S_2 has equation $r = 2\cos\theta$. The condition that $x \geq 0$ translates as $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Thus

$$D = \{(r,\theta) \mid -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2},\ 2\cos\theta \leq r \leq 2\}.$$

2 marks (c) Using any method, compute the area of D.

Answer: π

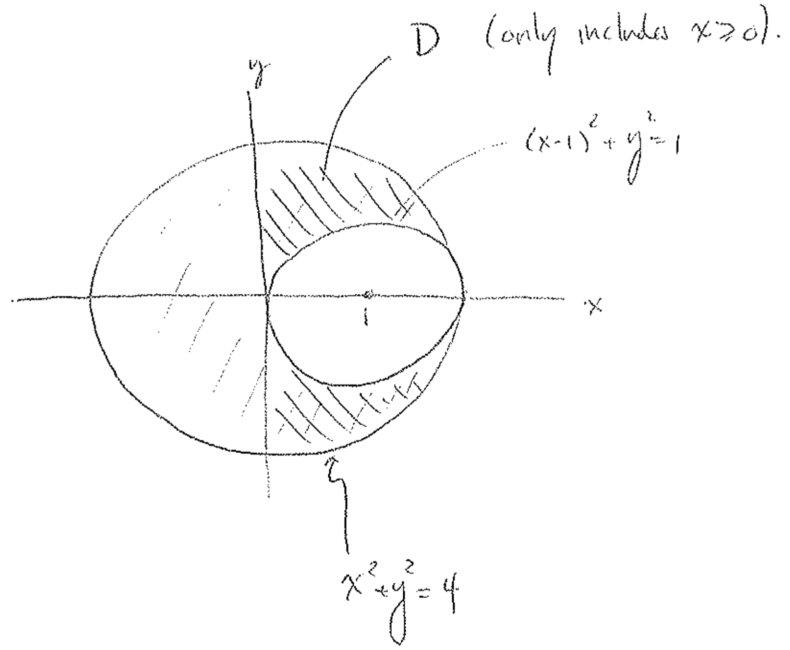
Solution: Solution 1:

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \int_{2\cos\theta}^{2} r\,dr\,d\theta = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}r^2\Big|_{2\cos\theta}^{2} d\theta = 2\pi - 2\int_0^{\frac{\pi}{2}} 2\cos^2\theta d\theta$$

$$= 2\pi - 2\int_0^{\frac{\pi}{2}} (1+\cos(2\theta))d\theta = 2\pi - 2[\theta + \frac{1}{2}\sin(2\theta)]_0^{\frac{\pi}{2}} = \pi.$$

Solution 2: The area is the area of half the big circle minus the area of the small circle, so $\frac{1}{2}\pi 2^2 - \pi 1^2 = \pi$.

(b)



5 marks 7. (a) Find the area of the portion of the cone $z^2 = x^2 + y^2$ lying between the planes $z = 2$ and $z = 3$.

Answer: $5\sqrt{2}\pi$

Solution: The top half of the cone is given by $z = \sqrt{x^2 + y^2}$, and

$$\frac{\partial z}{\partial x} = \frac{x}{z}, \quad \frac{\partial z}{\partial y} = \frac{y}{z}.$$

The portion of interest sits above the annulus D given by $2^2 \leq x^2 + y^2 \leq 3^2$, so

$$\begin{aligned}
A &= \int\int_D \left(1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right)^{1/2} dA \\
&= \int\int_D \left(1 + \left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2\right)^{1/2} dA \\
&= \int\int_D \left(\frac{x^2 + y^2 + z^2}{z^2}\right)^{1/2} dA \\
&= \int\int_D \left(\frac{2z^2}{z^2}\right)^{1/2} dA = \sqrt{2}\int\int_D dA \\
&= \sqrt{2}\int_0^{2\pi}\int_2^3 r\,dr\,d\theta = \sqrt{2}(2\pi)\frac{1}{2}r^2|_2^3 = 5\sqrt{2}\pi.
\end{aligned}$$

Or instead, at the end, use $\int\int_D dA$ equals the area of D which is $\pi 3^2 - \pi 2^2 = 5\pi$.

5 marks (b) Find the centre of mass of a triangular lamina with vertices $(0,0)$, $(1,0)$ and $(0,1)$ and density $\rho(x,y) = x+y$.

Answer: $M = \frac{1}{3}$, $(\bar{x},\bar{y}) = (\frac{3}{8}, \frac{3}{8})$.

Solution: The mass is

$$M = \int_0^1 \int_0^{1-x} (x+y)dydx = \int_0^1 \left(xy + \frac{1}{2}y^2\right)\Big|_{y=0}^{y=1-x} dx$$
$$= \int_0^1 \left(\frac{1}{2} - \frac{1}{2}x^2\right) dx = \left(\frac{1}{2}x - \frac{1}{6}x^3\right)\Big|_0^1 = \frac{1}{3}.$$

Also,

$$M\bar{x} = \int_0^1 \int_0^{1-x} x(x+y)dydx = \int_0^1 \left(x^2y + \frac{1}{2}xy^2\right)\Big|_{y=0}^{y=1-x} dx$$
$$= \int_0^1 \left(\frac{1}{2}x - \frac{1}{2}x^3\right) dx = \left(\frac{1}{4}x^2 - \frac{1}{8}x^4\right)\Big|_0^1 = \frac{1}{8}.$$

Therefore $\bar{x} = (1/8)/(1/3) = \frac{3}{8}$. By symmetry, $\bar{y} = \bar{x}$.

4 marks 8. (a) Let E be the upper half space $z \geq 0$. Evaluate

$$I = \int\int\int_E \frac{e^{-(x^2+y^2+z^2)}}{\sqrt{x^2+y^2+z^2}}dV.$$

Answer: π.

Solution: We use spherical coordinates:

$$\begin{aligned} I &= \int_0^{2\pi}\int_0^{\pi/2}\int_0^{\infty} e^{-\rho^2}\frac{1}{\rho}\rho^2 \sin\phi d\rho d\phi d\theta \\ &= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi/2} \sin\phi d\phi\right)\left(\int_0^{\infty} e^{-\rho^2}\rho d\rho\right) \\ &= (2\pi)(1)\left(\frac{1}{2}\right) = \pi. \end{aligned}$$

5 marks (b) Let E be the solid region inside the sphere $x^2+y^2+z^2=5$ and above the paraboloid $z = 2(x^2+y^2)$. Evaluate the integral $I = \int\int\int_E z dV$.

Answer: $\frac{19}{12}\pi$

Solution: We use cylindrical coordinates:

$$\begin{aligned} I &= \int_0^{2\pi}\int_0^1\int_{2r^2}^{\sqrt{5-r^2}} z dz r dr d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^1 (5-r^2-4r^4) r dr d\theta \\ &= \frac{1}{2}\int_0^{2\pi}\left(\frac{5}{2}r^2 - \frac{1}{4}r^4 - \frac{4}{6}r^6\right)\Big|_0^1 d\theta \\ &= \frac{1}{2}\int_0^{2\pi}\frac{19}{12}d\theta = \frac{19}{12}\pi. \end{aligned}$$

5 marks (c) Completely reverse the order of integration and evaluate $I = \int_0^1 \int_x^1 \int_y^1 e^{z^3}\, dz dy dx$.

Answer: $\frac{1}{6}(e-1)$

Solution: The region of integration is $E = \{(x, y, z) : 0 \le x \le y \le z \le 1\}$, and

$$I = \int\int\int_E e^{z^3}\, dV = \int_0^1 \int_0^z \int_0^y e^{z^3}\, dx dy dz = \int_0^1 \int_0^z y e^{z^3}\, dy dz$$
$$= \frac{1}{2} \int_0^1 z^2 e^{z^3}\, dz = \frac{1}{6} e^{z^3}|_0^1 = \frac{1}{6}(e-1).$$