Final Examination — December 12, 2017 Duration: 2.5 hours

This test has ***10 questions*** *on* ***10 pages****, for a total of 80 points.*

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- Read all the questions carefully before starting to work. Unless otherwise indicated, give complete arguments and explanations for all your calculations as answers without justification will not be marked.
- Continue on a blank page if you run out of space, and **indicate this clearly on the original page.**
- This is a closed-book examination. **No aids of any kind are allowed**, including: documents, cheat sheets, electronic devices of any kind (including calculators, phones, etc.)

First Name: Solutions Last Name: ____________

Student-No: ____________ Section: ____________

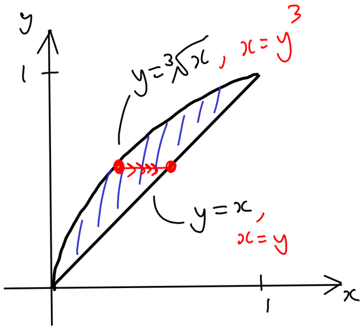
Signature: ____________

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	6	6	15	6	6	8	6	4	8	15	80
Score:											

6 marks 1. Sketch the region of integration and evaluate $\int_0^1 \int_x^{\sqrt[3]{x}} e^{x/y}\, \mathrm{d}y\, \mathrm{d}x$.

Answer: $\frac{1}{2}$

Solution:



Switch the order of integration. The domain is the bounded region between the curves $y = x$ and $y = \sqrt[3]{x}$. The latter is also $y^3 = x$, and so switching the order of integration, the iterated integral becomes

$$\int_0^1 \int_{y^3}^{y} e^{x/y}\, dx\, dy = \int_0^1 \left(ye^{x/y} \Big|_{x=y^3}^{x=y} \right) dy$$

$$= \int (ey - ye^{y^2})\, dy$$

$$= \left(e\frac{y^2}{2} - \frac{1}{2}e^{y^2} \right) \Big|_0^1 = \frac{1}{2}.$$

6 marks

2. A herd of goats lives in a circular pen. They want to know the area of their pen. They pace the diameter and find it to be $d = 10$ paces plus or minus 2 paces (they're not great at counting). The goats know that area is proportional to radius squared; thus they have derived the area formula $A = pr^2$. Goats don't know about "π" but they have experimentally determined that $p = 3$ plus or minus 0.5.

   Use the total differential to estimate the error they should expect to make in computing A based on these measurements. Give units.

Answer: 42.5 paces^2

Solution: Let $A(p, d) = p(d/2)^2$. Total differential:

$$\mathrm{d}A = \frac{\partial A}{\partial p}\,\mathrm{d}p + \frac{\partial A}{\partial d}\,\mathrm{d}d.$$

Now $\frac{\partial A}{\partial p} = (d/2)^2$ and $\frac{\partial A}{\partial d} = p(d/2)$, so $\mathrm{d}A = (10/2\,\text{paces})^2 \cdot 0.5 + 3 \cdot (10/2\,\text{paces}) \cdot (2\,\text{paces}) = 25/, \text{paces}^2 + 30\,\text{paces}^2 = 42.5\,\text{paces}^2$.

6 marks

3. (a) Consider the function $f_\alpha(x, y) = x^2 + y^2 - \alpha xy$ with $\alpha \geq 0$ being a nonnegative number. Give the gradient $\nabla f_\alpha(x, y)$.

Answer: $\nabla f_\alpha(x, y) = \langle 2x - \alpha y, 2y - \alpha x\rangle$

(b) What equations must be solved to find the critical points?

Answer: $2x - \alpha y = 0$ and $2y - \alpha x = 0$

Solution: $2x - \alpha y = 0$ and $2y - \alpha x = 0$

(c) Carefully describe the set of critical points (x, y).

Solution: $(0, 0)$ is always a solution. Unless $\alpha \neq 2$ it is the only solution.
If $\alpha = 2$, then any points (x, y) with $x = y$ is a solution.

4 marks

(d) For $\alpha = 2$, what can you conclude about the critical point(s) using only the second derivative test?

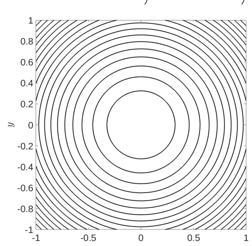
Solution: Note $f_{xx} = 2, f_{xy} = -\alpha, f_{yy} = 2$. So $D(x, y) = 4 - \alpha^2$. When $\alpha = 2$, $D(0, 0) = 0$ and the test is inconclusive.

(e) For what value(s) of α do we have a saddle point?

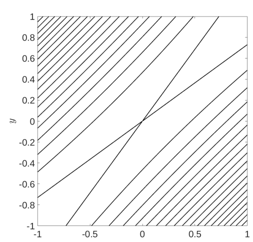
Answer: Want $D(0, 0) = 4 - \alpha^2 < 0$ so we need $\alpha > 2$.

5 marks

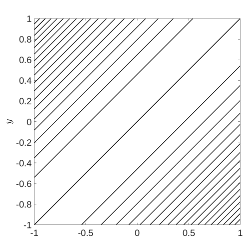
(f) Associate the contour plots for f_α with the corresponding values of $\alpha = 20$, $\alpha = 2.1$, $\alpha = 2$, $\alpha = 1$, and $\alpha = 0$.



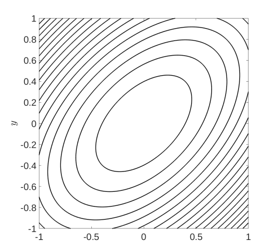
Answer: $\alpha = 0$



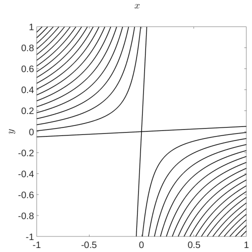
Answer: $\alpha = 2.1$



Answer: $\alpha = 2$



Answer: $\alpha = 1$



Answer: $\alpha = 20$

3 marks 4. (a) Consider the equation $F(x, y, z) = xy^2z^4 - \ln z - 1$. Compute the three partial derivatives F_x, F_y and F_z.

Answer: $\frac{\partial F}{\partial x} = y^2z^3$

Answer: $\frac{\partial F}{\partial y} = 2xyz^3$

Answer: $\frac{\partial F}{\partial z} = 4xy^2z^3 - 1/z$

3 marks (b) A surface is defined implicitly by $F(x, y, z) = 0$. Find an equation for the tangent plane to the surface at the point $(1, 1, 1)$.

Answer: $x + 2y + 3z = 6$

Solution: The equation of the tangent plane at (x_0, y_0, z_0) is given by

$$\vec{\nabla} F(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

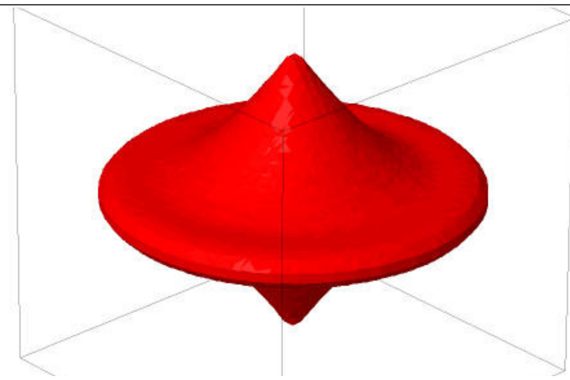
We obtain that $\vec{\nabla} F(1, 1, 1) = (1, 2, 3)$ and thus the equation of the tangent plane is reduced to $(1, 2, 3) \cdot (x - 1, y - 1, z - 1) = 0$. We deduce that the required equation is $x - 1 + 2(y - 1) + 3(z - 1) = 0$ or $x + 2y + 3z = 6$

3 marks 5. (a) Consider the function $G(x, y, z) = \left(\sqrt{x^2 + y^2} - 3\right)^3 + z^2 - 1$. Compute the partial derivatives of G.

Answer: $\frac{\partial G}{\partial x} = \frac{3x(\sqrt{x^2+y^2}-3)^2}{\sqrt{x^2+y^2}}$

Answer: $\frac{\partial G}{\partial x} = \frac{3y(\sqrt{x^2+y^2}-3)^2}{\sqrt{x^2+y^2}}$

Answer: $\frac{\partial G}{\partial z} = 2z$



3 marks (b) Let (a, b, c) be a point on the surface defined implicitly by $G(x, y, z) = 0$. Let T be the tangent plane to the surface at the point (a, b, c). Find the set of points (a, b, c) such that T does not intersect the z-axis. Describe in words what set the points define.

Answer: $a^2 + b^2 = 16$ and $c = 0$, a circle in the xy-plane.

Solution: The normal vector to the tangent plane at (a, b, c) is given by the gradient at that point, that is $\vec{\nabla} G(a, b, c)$. Therefore in order not to intersect the z-axis the third coordinate of the gradient must be zero, that is, $2z = 0$ yielding $z = 0$. It suffices thus to replace $z = 0$ in the formula of the surface, we obtain $(\sqrt{x^2 + y^2} - 3)^3 = 1$, yielding the solution $x^2 + y^2 = 16$. That is, a circle of radius 4.

8 marks 6. (a) State the definition of continuity of a function $f(x, y)$ at (a, b).

Solution:

$$\lim_{(x,y)\to(a,b)} f(x, y)$$

exists for all paths approaching (a, b). The limit must also be equal to $f(a, b)$.

(b) Consider the function $f(x, y) = \begin{cases} \frac{xy-y^2}{y^2+x^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$

Is f continuous at $(0, 0)$? Explain your answer.

Answer: No, different paths give different answers.

Solution:

Compute the limits along $y = mx$.

Along $y = mx$: $\lim_{x\to0} f(x, mx) = \lim_{x\to0} \frac{xmx-m^2x^2}{m^2x^2+x^2} = \lim_{x\to0} \frac{x^2(m-m^2)}{x^2(m^2+1)} = \frac{m-m^2}{m^2+1}$

These are different values for different m.

Or, could just check two specific paths. Along $y = 0$, $f(x, 0) = 0$. But along $x = 0$, $\lim_{y\to0} f(0, y) = \lim_{y\to0} -\frac{y^2}{y^2} = -1$

6 marks

7. (a) Let E be the region of $\mathbb{R}^3$ where $x^2+y^2+z^2 \le 1$, $x \ge 0$, $y \ge 0$, and $z \ge 0$. Write the region in spherical coordinates.

Solution:

$$x = \rho\sin\phi\cos\theta, y = \rho\sin\phi\sin\theta, z = \rho\cos\phi$$

$$E = \{0 \le \rho \le 1, 0 \le \phi \le \frac{\pi}{2},\ 0 \le \theta \le \frac{\pi}{2}$$

(b) Evaluate the triple integral $\displaystyle\iiint\limits_E \frac{1}{x^2+y^2+z^2}\,\mathrm{d}V$.

Answer: $\frac{\pi}{2}$

Solution:

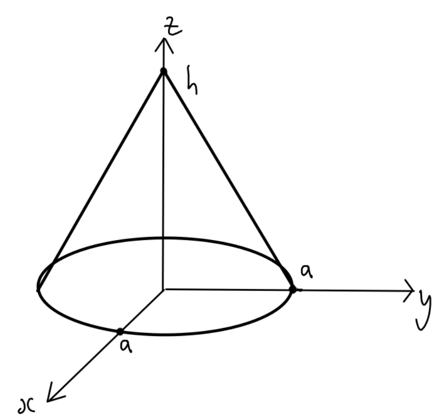
$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 \frac{1}{\rho^2}\,\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi = \left(\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\right)\left(\int_0^{\frac{\pi}{2}}\sin\phi\,\mathrm{d}\phi\right)\left(\int_0^1\mathrm{d}\rho\right)$$
$$= \left(\frac{\pi}{2}\right)\left(-\cos\theta\Big|_0^{\frac{\pi}{2}}\right)(1) = \frac{\pi}{2}$$

Old solution for $1+\rho^2$ in denom:

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 \frac{1}{1+\rho^2}\,\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi = \left(\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\right)\left(\int_0^{\frac{\pi}{2}}\sin\phi\,\mathrm{d}\phi\right)\left(\int_0^1\frac{\rho^2}{1+\rho^2}\,\mathrm{d}\rho\right)$$
$$= \dots$$

4 marks

8. Find the formula $z = f(x,y)$ for the surface shown in the figure, i.e., the top surface of a circular right cone with height h and base radius a. Also find the cylindrical formula $z = g(r,\theta)$.



Answer: $f(x,y) = h - \frac{h}{a}\sqrt{x^2+y^2}$

Answer: $g(r,\theta) = h - \frac{h}{a}r$

8 marks 9. Evaluate the triple iterated integral $\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{2}\frac{1}{1+x^2+y^2}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x.$

Answer: $\pi\ln(10)$

Solution: The region is a portion of a right circular cylinder oriented along the z-axis. It is easiest to view this as a Type 1 region and then to use cylindrical coordinates. The region of integration in the (x, y)-plane is the upper half of the disk of radius 3 centred at the origin. Note that the integrand does not depend on z. So, in cylindrical coordinates, the integral becomes

$$\int_0^2\int_0^\pi\int_0^3\frac{1}{1+r^2}\,rdr\;d\theta\;dz.$$

We evaluate this as

$$\begin{aligned}\int_0^2\int_0^\pi\int_0^3\frac{1}{1+r^2}\,r\,\mathrm{d}r\;\;\mathrm{d}\theta\;\;\mathrm{d}z &= \left(\int_0^2 dz\right)\left(\int_0^\pi d\theta\right)\left(\int_0^3\frac{1}{1+r^2}\,rdr\right)\\ &= 2\cdot\pi\cdot\frac{1}{2}\ln(10) = \pi\ln(10).\end{aligned}$$

3 marks 10. (a) State the integral formula for the surface area of an explicit surface $z = f(x, y)$ above region D.

Answer: $\iint_D \sqrt{1 + f_x^2 + f_y^2}\, \mathrm{d}A$

3 marks (b) Consider a surface $z = f(x, y)$. Give a formula for $\hat{n}$, the *unit* upward-facing normal vector to the surface. Hint: your answer should be a 3-vector.

Answer: $\hat{n} = \langle -f_x, -f_y, 1\rangle / \sqrt{f_x^2 + f_y^2 + 1}$

Solution: Let $F(x, y, z) = f(x, y) - z$ then $\vec{n} = \nabla F = \langle f_x, f_y, -1\rangle$.
For $\hat{n}$, start with $\langle -f_x, -f_y, 1\rangle$ (nonnegative in last position). Then normalize.

3 marks (c) Give an expression for $\cos\theta$, where θ is the angle between $\hat{n}$ and the z-axis.

Answer: $\cos\theta = \frac{1}{\sqrt{f_x^2+f_y^2+1}}$

Solution: $\cos\theta = \hat{n} \cdot \langle 0, 0, 1\rangle = \frac{0+0+1}{\sqrt{f_x^2+f_y^2+1}}$

3 marks (d) Grass—for feeding goats—is growing on a surface $z = f(x, y)$. The grass does not grow evenly; its density is given by a function $g(x, y)$ (units: kg/m^2). Write an integral for the total amount of grass contained on the surface above a region D of the xy-plane. Hint: in a small patch, the amount of grass depends both on the density and the area of the patch.

Answer: $\iint_D g(x, y)\sqrt{1 + f_x^2 + f_y^2}\, \mathrm{d}A$.

3 marks (e) For reasons of drainage, suppose grass density is proportional to the cosine of the angle the surface normal makes with the vector $\langle 0, 0, 1\rangle$. That is, $g(x, y) = \rho_0 \cos\theta$.

A goat lives in a rectangular pen $(x, y) \in [0, 3] \times [-2, 2]$. Determine the total mass of grass in her pen.

Answer: $12\rho_0$ kg

Solution: Epic cancellation ensues.