

Final Examination — December 12, 2017 Duration: 2.5 hours*This test has 10 questions on 10 pages, for a total of 80 points.*

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- Read all the questions carefully before starting to work. Unless otherwise indicated, give complete arguments and explanations for all your calculations as answers without justification will not be marked.
- Continue on a blank page if you run out of space, and **indicate this clearly on the original page.**
- This is a closed-book examination. **No aids of any kind are allowed**, including: documents, cheat sheets, electronic devices of any kind (including calculators, phones, etc.)

First Name: Solutions Last Name: _____

Student-No: _____ Section: _____

Signature: _____

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	6	6	15	6	6	8	6	4	8	15	80
Score:											

Student Conduct during Examinations

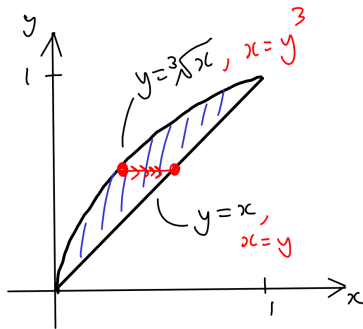
- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
- Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
- Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
 - speaking or communicating with other examination candidates, unless otherwise authorized;
 - purposely exposing written papers to the view of other examination candidates or imaging devices;
 - purposely viewing the written papers of other examination candidates;
 - using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
 - using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)—(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

6 marks

1. Sketch the region of integration and evaluate $\int_0^1 \int_x^{\sqrt[3]{x}} e^{x/y} dy dx$.

Answer: $\frac{1}{2}$

Solution:



Switch the order of integration. The domain is the bounded region between the curves $y = x$ and $y = \sqrt[3]{x}$. The latter is also $y^3 = x$, and so switching the order of integration, the iterated integral becomes

$$\begin{aligned} \int_0^1 \int_{y^3}^y e^{x/y} dx dy &= \int_0^1 \left(y e^{x/y} \Big|_{x=y^3}^{x=y} \right) dy \\ &= \int (ey - ye^{y^2}) dy \\ &= \left(e \frac{y^2}{2} - \frac{1}{2} e^{y^2} \right) \Big|_0^1 = \frac{1}{2}. \end{aligned}$$

6 marks

2. A herd of goats lives in a circular pen. They want to know the area of their pen. They pace the diameter and find it to be $d = 10$ paces plus or minus 2 paces (they're not great at counting). The goats know that area is proportional to radius squared; thus they have derived the area formula $A = pr^2$. Goats don't know about " π " but they have experimentally determined that $p = 3$ plus or minus 0.5.

Use the total differential to estimate the error they should expect to make in computing A based on these measurements. Give units.

Answer: 42.5 paces ²

Solution: Let $A(p, d) = p(d/2)^2$. Total differential:

$$dA = \frac{\partial A}{\partial p} dp + \frac{\partial A}{\partial d} dd.$$

Now $\frac{\partial A}{\partial p} = (d/2)^2$ and $\frac{\partial A}{\partial d} = p(d/2)$, so $dA = (10/2 \text{ paces})^2 \cdot 0.5 + 3 \cdot (10/2 \text{ paces}) \cdot (2 \text{ paces}) = 25/2 \text{ paces}^2 + 30 \text{ paces}^2 = 42.5 \text{ paces}^2$.

6 marks

3. (a) Consider the function $f_\alpha(x, y) = x^2 + y^2 - \alpha xy$ with $\alpha \geq 0$ being a nonnegative number. Give the gradient $\nabla f_\alpha(x, y)$.

Answer: $\nabla f_\alpha(x, y) = \langle 2x - \alpha y, 2y - \alpha x \rangle$

- (b) What equations must be solved to find the critical points?

Answer: $2x - \alpha y = 0$ and $2y - \alpha x = 0$

Solution: $2x - \alpha y = 0$ and $2y - \alpha x = 0$

- (c) Carefully describe the set of critical points (x, y) .

Solution: $(0, 0)$ is always a solution. Unless $\alpha \neq 2$ it is the only solution. If $\alpha = 2$, then any points (x, y) with $x = y$ is a solution.

4 marks

- (d) For $\alpha = 2$, what can you conclude about the critical point(s) using only the second derivative test?

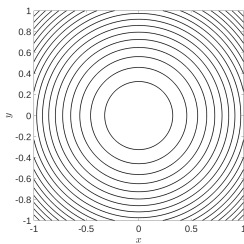
Solution: Note $f_{xx} = 2, f_{xy} = -\alpha, f_{yy} = 2$. So $D(x, y) = 4 - \alpha^2$. When $\alpha = 2$, $D(0, 0) = 0$ and the test is inconclusive.

- (e) For what value(s) of α do we have a saddle point?

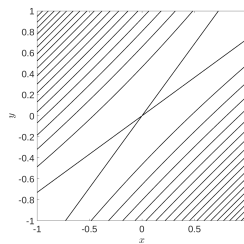
Answer: Want $D(0, 0) = 4 - \alpha^2 < 0$ so we need $\alpha > 2$.

5 marks

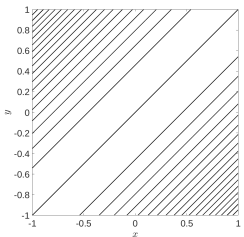
- (f) Associate the contour plots for f_α with the corresponding values of $\alpha = 20, \alpha = 2.1, \alpha = 2, \alpha = 1$, and $\alpha = 0$.



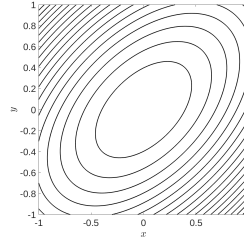
Answer:
 $\alpha = 0$



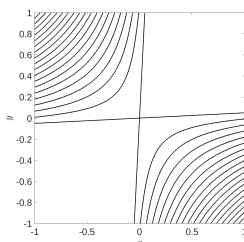
Answer:
 $\alpha = 2.1$



Answer:
 $\alpha = 2$



Answer:
 $\alpha = 1$



Answer:
 $\alpha = 20$

3 marks

4. (a) Consider the equation $F(x, y, z) = xy^2z^4 - \ln z - 1$. Compute the three partial derivatives F_x , F_y and F_z .

$$\text{Answer: } \frac{\partial F}{\partial x} = y^2z^3$$

$$\text{Answer: } \frac{\partial F}{\partial y} = 2xyz^3$$

$$\text{Answer: } \frac{\partial F}{\partial z} = 4xy^2z^3 - 1/z$$

3 marks

- (b) A surface is defined implicitly by $F(x, y, z) = 0$. Find an equation for the tangent plane to the surface at the point $(1, 1, 1)$.

$$\text{Answer: } x + 2y + 3z = 6$$

Solution: The equation of the tangent plane at (x_0, y_0, z_0) is given by

$$\vec{\nabla} F(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

We obtain that $\vec{\nabla} F(1, 1, 1) = (1, 2, 3)$ and thus the equation of the tangent plane is reduced to $(1, 2, 3) \cdot (x - 1, y - 1, z - 1) = 0$. We deduce that the required equation is $x - 1 + 2(y - 1) + 3(z - 1) = 0$ or $x + 2y + 3z = 6$

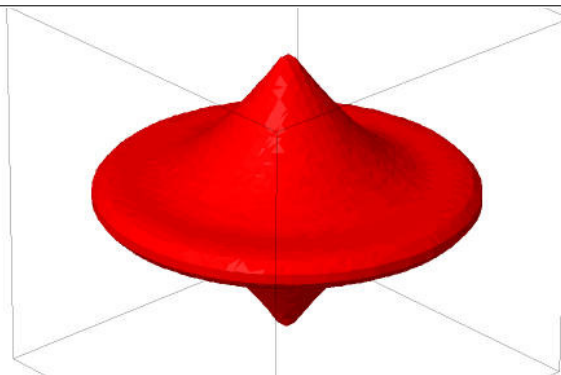
3 marks

5. (a) Consider the function $G(x, y, z) = \left(\sqrt{x^2 + y^2} - 3\right)^3 + z^2 - 1$. Compute the partial derivatives of G .

Answer: $\frac{\partial G}{\partial x} = \frac{3x(\sqrt{x^2 + y^2} - 3)^2}{\sqrt{x^2 + y^2}}$

Answer: $\frac{\partial G}{\partial y} = \frac{3y(\sqrt{x^2 + y^2} - 3)^2}{\sqrt{x^2 + y^2}}$

Answer: $\frac{\partial G}{\partial z} = 2z$



3 marks

- (b) Let (a, b, c) be a point on the surface defined implicitly by $G(x, y, z) = 0$. Let T be the tangent plane to the surface at the point (a, b, c) . Find the set of points (a, b, c) such that T does not intersect the z -axis. Describe in words what set the points define.

Answer: $a^2 + b^2 = 16$ and $c = 0$, a circle in the xy -plane.

Solution: The normal vector to the tangent plane at (a, b, c) is given by the gradient at that point, that is $\vec{\nabla}G(a, b, c)$. Therefore in order not to intersect the z -axis the third coordinate of the gradient must be zero, that is, $2z = 0$ yielding $z = 0$. It suffices thus to replace $z = 0$ in the formula of the surface, we obtain $(\sqrt{x^2 + y^2} - 3)^3 = 1$, yielding the solution $x^2 + y^2 = 16$. That is, a circle of radius 4.

8 marks

6. (a) State the definition of continuity of a function $f(x, y)$ at (a, b) .

Solution:

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y)$$

exists for all paths approaching (a, b) . The limit must also be equal to $f(a, b)$.

- (b) Consider the function $f(x, y) = \begin{cases} \frac{xy-y^2}{y^2+x^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$

Is f continuous at $(0, 0)$? Explain your answer.

Answer: No, different paths give different answers.

Solution:

Compute the limits along $y = mx$.

Along $y = mx$: $\lim_{x \rightarrow 0} f(x, mx) = \lim_{x \rightarrow 0} \frac{mx-m^2x^2}{m^2x^2+x^2} = \lim_{x \rightarrow 0} \frac{x^2(m-m^2)}{x^2(m^2+1)} = \frac{m-m^2}{m^2+1}$
 These are different values for different m .

Or, could just check two specific paths. Along $y = 0$, $f(x, 0) = 0$. But along $x = 0$, $\lim_{y \rightarrow 0} f(0, y) = \lim_{y \rightarrow 0} -\frac{y^2}{y^2} = -1$

6 marks

7. (a) Let E be the region of \mathbb{R}^3 where $x^2 + y^2 + z^2 \leq 1$, $x \geq 0$, $y \geq 0$, and $z \geq 0$. Write the region in spherical coordinates.

Solution:

$$x = \rho \sin \phi \cos \theta, y = \rho \sin \phi \sin \theta, z = \rho \cos \phi$$

$$E = \{0 \leq \rho \leq 1, 0 \leq \phi \leq \frac{\pi}{2}, 0 \leq \theta \leq \frac{\pi}{2}\}$$

- (b) Evaluate the triple integral $\iiint_E \frac{1}{x^2 + y^2 + z^2} dV$.

Answer: $\frac{\pi}{2}$ **Solution:**

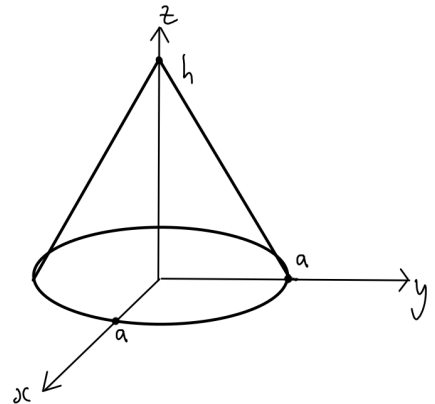
$$\begin{aligned} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^1 \frac{1}{\rho^2} \rho^2 \sin \phi d\rho d\theta d\phi &= \left(\int_0^{\frac{\pi}{2}} d\theta \right) \left(\int_0^{\frac{\pi}{2}} \sin \phi d\phi \right) \left(\int_0^1 d\rho \right) \\ &= \left(\frac{\pi}{2} \right) \left(-\cos \theta \Big|_0^{\frac{\pi}{2}} \right) (1) = \frac{\pi}{2} \end{aligned}$$

Old solution for $1 + \rho^2$ in denom:

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^1 \frac{1}{1 + \rho^2} \rho^2 \sin \phi d\rho d\theta d\phi &= \left(\int_0^{\frac{\pi}{2}} d\theta \right) \left(\int_0^{\frac{\pi}{2}} \sin \phi d\phi \right) \left(\int_0^1 \frac{\rho^2}{1 + \rho^2} d\rho \right) \\ &= \dots \end{aligned}$$

4 marks

8. Find the formula $z = f(x, y)$ for the surface shown in the figure, i.e., the top surface of a circular right cone with height h and base radius a . Also find the cylindrical formula $z = g(r, \theta)$.

Answer: $f(x, y) = h - \frac{h}{a} \sqrt{x^2 + y^2}$ Answer: $g(r, \theta) = h - \frac{h}{a} r$

8 marks

9. Evaluate the triple iterated integral $\int_{-3}^3 \int_0^{\sqrt{9-x^2}} \int_0^2 \frac{1}{1+x^2+y^2} dz dy dx$.

Answer: $\pi \ln(10)$

Solution: The region is a portion of a right circular cylinder oriented along the z -axis. It is easiest to view this as a Type 1 region and then to use cylindrical coordinates. The region of integration in the (x, y) -plane is the upper half of the disk of radius 3 centred at the origin. Note that the integrand does not depend on z . So, in cylindrical coordinates, the integral becomes

$$\int_0^2 \int_0^\pi \int_0^3 \frac{1}{1+r^2} r dr d\theta dz.$$

We evaluate this as

$$\begin{aligned} \int_0^2 \int_0^\pi \int_0^3 \frac{1}{1+r^2} r dr d\theta dz &= \left(\int_0^2 dz \right) \left(\int_0^\pi d\theta \right) \left(\int_0^3 \frac{1}{1+r^2} r dr \right) \\ &= 2 \cdot \pi \cdot \frac{1}{2} \ln(10) = \pi \ln(10). \end{aligned}$$

- 3 marks 10. (a) State the integral formula for the surface area of an explicit surface $z = f(x, y)$ above region D .

$$\text{Answer: } \iint_D \sqrt{1 + f_x^2 + f_y^2} \, dA$$

- 3 marks (b) Consider a surface $z = f(x, y)$. Give a formula for \hat{n} , the *unit* upward-facing normal vector to the surface. Hint: your answer should be a 3-vector.

$$\text{Answer: } \hat{n} = \langle -f_x, -f_y, 1 \rangle / \sqrt{f_x^2 + f_y^2 + 1}$$

Solution: Let $F(x, y, z) = f(x, y) - z$ then $\vec{n} = \nabla F = \langle f_x, f_y, -1 \rangle$.

For \hat{n} , start with $\langle -f_x, -f_y, 1 \rangle$ (nonnegative in last position). Then normalize.

- 3 marks (c) Give an expression for $\cos \theta$, where θ is the angle between \hat{n} and the z -axis.

$$\text{Answer: } \cos \theta = \frac{1}{\sqrt{f_x^2 + f_y^2 + 1}}$$

$$\text{Solution: } \cos \theta = \hat{n} \cdot \langle 0, 0, 1 \rangle = \frac{0+0+1}{\sqrt{f_x^2 + f_y^2 + 1}}$$

- 3 marks (d) Grass—for feeding goats—is growing on a surface $z = f(x, y)$. The grass does not grow evenly; its density is given by a function $g(x, y)$ (units: kg/m²). Write an integral for the total amount of grass contained on the surface above a region D of the xy -plane. Hint: in a small patch, the amount of grass depends both on the density and the area of the patch.

$$\text{Answer: } \iint_D g(x, y) \sqrt{1 + f_x^2 + f_y^2} \, dA.$$

- 3 marks (e) For reasons of drainage, suppose grass density is proportional to the cosine of the angle the surface normal makes with the vector $\langle 0, 0, 1 \rangle$. That is, $g(x, y) = \rho_0 \cos \theta$. A goat lives in a rectangular pen $(x, y) \in [0, 3] \times [-2, 2]$. Determine the total mass of grass in her pen.

$$\text{Answer: } 12\rho_0 \text{ kg}$$

Solution: Epic cancellation ensues.