5 marks

1. (a) Describe the equation of the plane which contains the points $A = (0,0,1)$, $B = (1,1,3)$ and $C = (1,-3,1)$ in the form $ax + y + cz + d = 0$. Note that we ask that the coefficient next to y is 1.

Solution: Use A as a base point and form the vectors $\vec{AB} = \langle 1,1,3\rangle - \langle 0,0,1\rangle = \langle 1,1,2\rangle$ and $\vec{AC} = \langle 1,-3,1\rangle - \langle 0,0,1\rangle = \langle 1,-3,0\rangle$. Use both vectors to compute a normal vector to the plane using the cross product $\vec{AB}\times\vec{AC}$. Now, $\vec{AB}\times\vec{AC} = \langle 6,2,-4\rangle$, so the solution is of the form $6x + 2y - 4z + d = 0$. Plugging any of the three points yields that $d = 4$. Dividing by 2 we get $3x + y - 2z + 2 = 0$.

Answer: $3x + y - 2z + 2 = 0$.

5 marks

(b) What angle does the plane from (a) form with the plane given by the equation $x - y = 0$? You may leave your answer in the form $\theta = \cos^{-1}(\cdot)$.

Solution: A vector which is normal to the plane $x - y = 0$ is $\vec{n} = (1,-1,0)$. A normal vector to the first plane is $\vec{m} = (3,1,-2)$. We can use the dot product formula to obtain that $\cos\theta = \frac{\vec{n}\cdot\vec{m}}{||\vec{n}||||\vec{m}||} = \frac{2}{\sqrt{2}\sqrt{14}}$.

Answer: $\theta = \cos^{-1}(\frac{1}{\sqrt{7}})$

5 marks

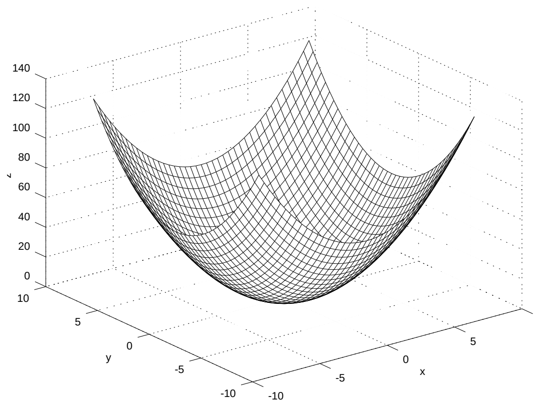
(c) The lines $\vec{l_1}$ and $\vec{l_2}$ are described by the equations $\vec{l_1}(t) = \langle 0,-2,1\rangle + t\langle 1,2,1\rangle$ and $\vec{l_2}(s) = \langle 2,2,3\rangle + s\langle 1,-1,2\rangle$. Give the equation of the line which has as a basepoint the intersection of $\vec{l_1}$ and $\vec{l_2}$ and is orthogonal to the plane spanned by $\vec{l_1}$ and $\vec{l_2}$.

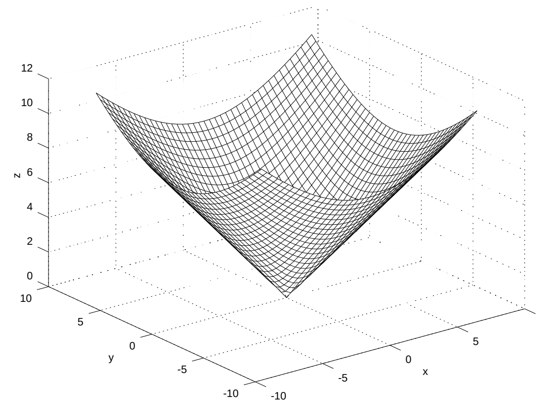
Solution: First we solve $\langle 0,-2,1\rangle + t\langle 1,2,1\rangle = \langle 2,2,3\rangle + s\langle 1,-1,2\rangle$. We obtain the equations $t = 2 + s$, $-2 + 2t = 2 - s$ and $1 + t = 3 + 2s$. Which admit the unique solution $s = 0$ and $t = 2$. Thus the intersection is $\langle 2,2,3\rangle$. A vector orthogonal to both lines is $\langle 1,2,1\rangle \times \langle 1,-1,2\rangle = \langle 5,-1,3\rangle$. Thus the equation is $\vec{l_3}(r) = \langle 2,2,3\rangle + r\langle 5,-1,3\rangle$.

Answer: $\vec{l_3}(r) = \langle 2,2,3\rangle + r\langle 5,-1,-3\rangle$.

5 marks

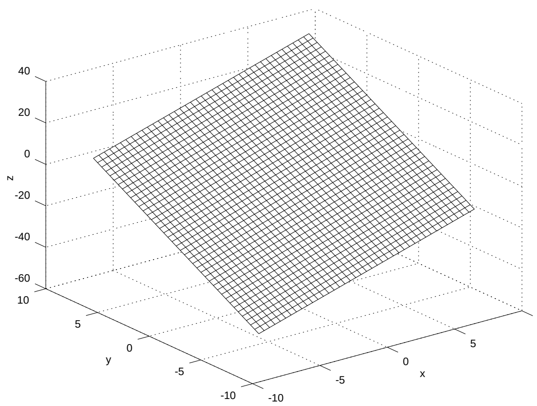
(d) Match the surfaces with the equations below. There is one equation which does not correspond to any surface.

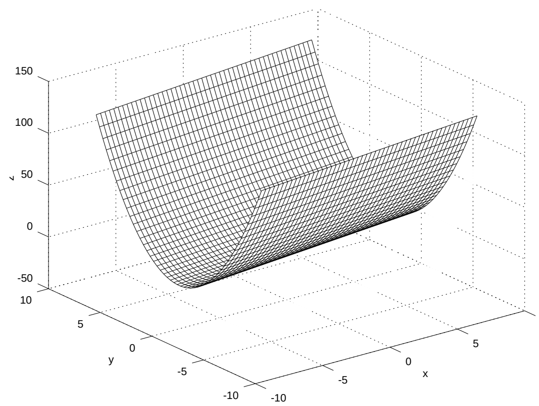




Answer: D

Answer: A





Answer: E

Answer: B

$(A)\quad z^2 = x^2 + y^2,$ $\qquad (B)\quad z = x + 2y^2,$ $\qquad (C)\quad x^2 + y^2 + z^2 = 1,$

$(D)\quad z = x^2 + y^2,$ $\qquad (E)\quad 2x + 3y - z = 4.$

2 marks

(e) (Bonus marks) Let $\vec{u} \neq \vec{0}$ be a fixed vector. Describe the set of vectors $\vec{v}$ which satisfy

$$||\vec{u} + \vec{v}|| = ||\vec{u}|| + ||\vec{v}||.$$

Solution: For $\vec{v} = C\vec{u}$ for some non-negative constant C.

Answer: For $\vec{v} = C\vec{u}$, $C \geq 0$.