The Lanford-Ruelle theorem for sofic groups

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UT Groups and Dynamics
November, 2022

Consider n particles with weights on $\mathbb{R}$ given by a function f

1	2	3	4	5	6	7	$\cdots$	n
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Which is the probability distribution $\mu = (\mu_1, \dots, \mu_n)$ on $\{1, \dots, n\}$ that maximizes entropy plus the integral of the weight?

$$\max_{\mu} \left(H(\mu) + \int f \mathrm{d}\mu \right) = \max_{\mu} \sum_{i=1}^{n} \left(-\mu_i \log(\mu_i) + f(i)\mu_i \right).$$

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Answer: Boltzmann's distribution.

$$\mu_k = \frac{\exp(f(k))}{\sum_{i=1}^{n} \exp(f(i))}.$$

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A subshift $X \subset A^{\Gamma}$ is of **finite type (SFT)** if there is $F \Subset \Gamma$ and $L \subset A^F$ such that $x \in X$ if and only if $gx \in \bigcup_{p \in L}[p]$ for every $g \in \Gamma$.

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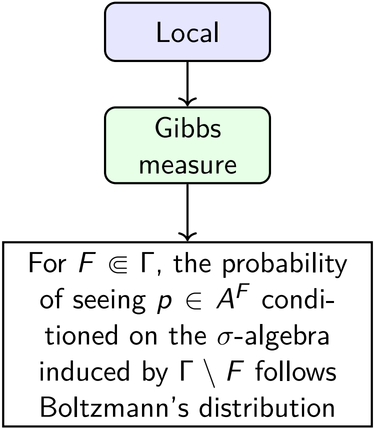

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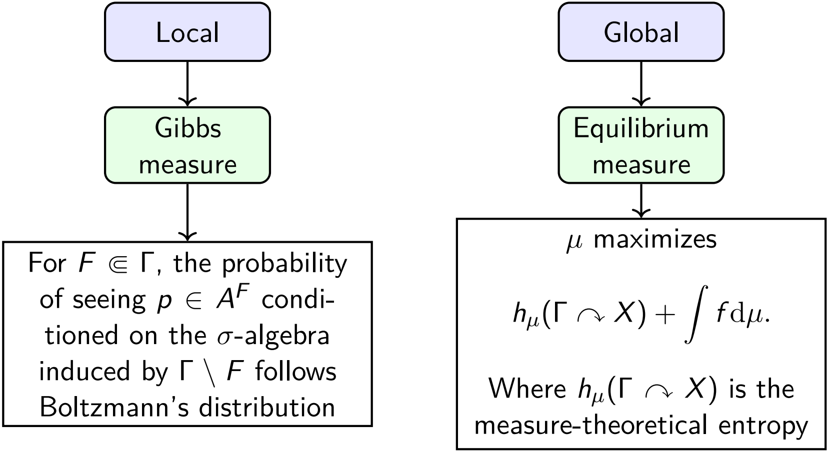

How are these two notions related?

Let $\Gamma = \mathbb{Z}^d$ and

1. A subshift $X \subset A^{\mathbb{Z}^d}$.
2. A sufficiently regular map $f\colon X \to \mathbb{R}$.
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Theorem: Dobrushin, 1968

If X is sufficiently mixing (D-mixing).
μ is Gibbs $\implies$ μ is equilibrium.

Today we'll present a version of the LR theorem on steroids.

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Theorem: B., Meyerovitch, 2021

Let Γ be a sofic group and Σ a sofic approximation sequence of Γ. Let $f : X \to \mathbb{R}$ be sufficiently regular. For every Γ-subshift which has TMP and $h_\Sigma(\Gamma \curvearrowright X) \geq 0$, every equilibrium measure μ is Gibbs.

Motivation

🦙 A **cellular automata** on $\Gamma \curvearrowright A^\Gamma$ is a continuous and Γ-equivariant map $\varphi \colon A^\Gamma \to A^\Gamma$

$$\varphi(gx) = g\varphi(x) \text{ for every } g \in \Gamma, x \in A^\Gamma.$$

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For every group Γ, every injective CA $\varphi \colon A^\Gamma \to A^\Gamma$ is surjective.

- Sofic groups satisfy the conjecture
- It is not even known if all groups are sofic. The conjecture is open.

Motivation

An abstract entropy theory on A^Γ is a map

$$(X, \mu) \mapsto h(\Gamma \curvearrowright X, \mu) \in [-\infty, \infty],$$

such that h is invariant under measurable dynamical isomorphism.

- $X \subset A^\Gamma$ is a subshift.
- μ is an invariant Borel probabily measure on A^Γ with support on X.

An abstract entropy theory satisfies the Lanford-Ruelle theorem for $f = 0$ if every measure μ such that

$$h(\Gamma \curvearrowright X, \mu) = \sup_\nu h(\Gamma \curvearrowright X, \nu),$$

is a Gibbs measure.

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- If φ is an injective endomorphism, then for every measure μ on A^Γ, $\Gamma \curvearrowright (A^\Gamma, \mu) \cong \Gamma \curvearrowright (\varphi(A^\Gamma), \varphi_*(\mu))$.

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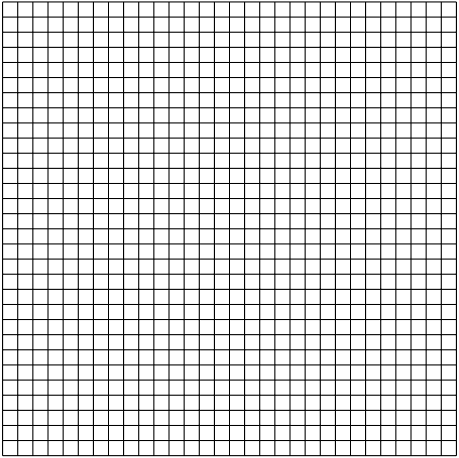
- If μ is uniform Bernoulli, then $h(\Gamma \curvearrowright A^\Gamma, \mu) = h(\Gamma \curvearrowright \varphi(A^\Gamma), \varphi_*(\mu))$ and thus $\varphi(A^\Gamma) = A^\Gamma$.

Gibbs's measure



A measure is Gibbs if it "locally follows Boltzmann's distribution".

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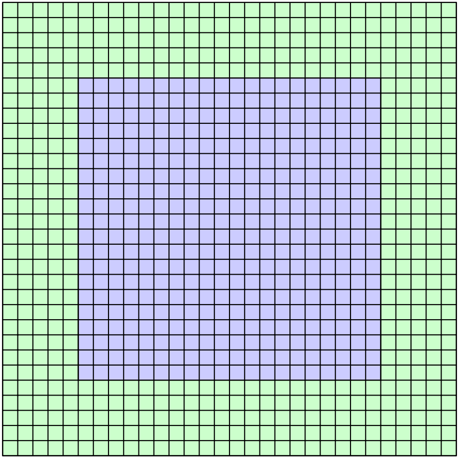

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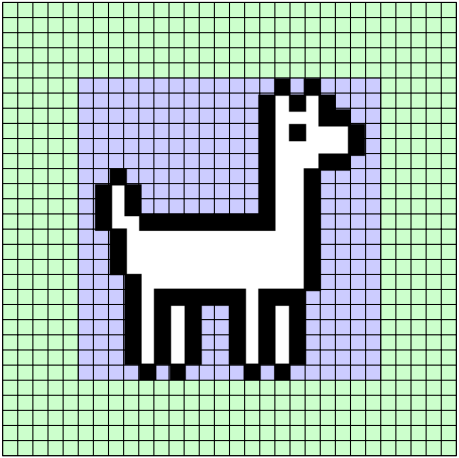

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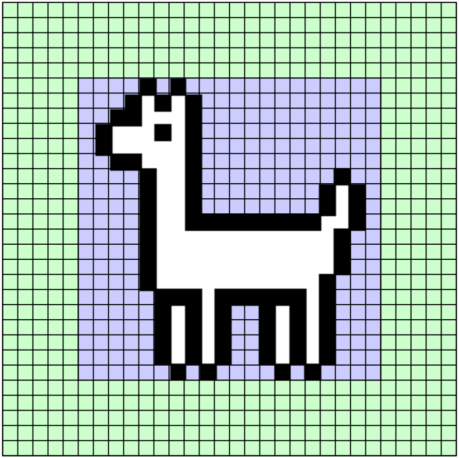

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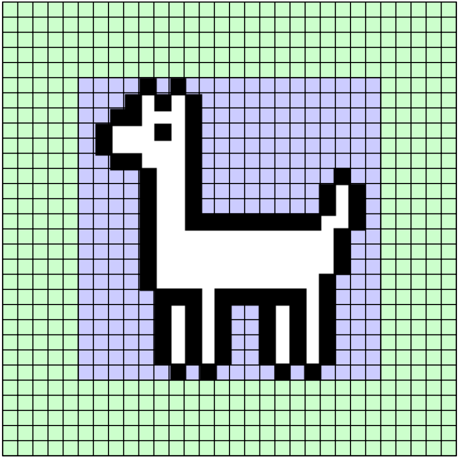

$\mu($$|$$)$ and $\mu($$|$$)$ follow Boltzmann's distribution.

Gibbs measure

$x, y \in A^\Gamma$ are **asymptotic** if there is $F \Subset \Gamma$ such that

$$x|_{\Gamma \setminus F} = y|_{\Gamma \setminus F}.$$

Denote by $\mathcal{T}(X)$ the equivalence relation of asymptotic pairs on X.

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Let us consider $f \colon X \to \mathbb{R}$ such that for every asymptotic pair $(x, y) \in \mathcal{T}(X)$

$$\Psi_f(x, y) = \sum_{g \in \Gamma} f(gy) - f(gx) \text{ is absolutely convergent.}$$

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Example: f is local (there is $F \Subset \Gamma$ such that $f(x) = f(y)$ when $x|_F = y|_F$).

Gibbs measure

Fix a subshift X. For $x \in A^\Gamma$ and $F \Subset \Gamma$, let $L_F(x)$ the set of patterns $p \in A^F$ which concatenated with $x|_{\Gamma \setminus F}$ give an element of X.

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A Borel probability measure μ in a subshift X is **Gibbs** with respect to $f : X \to \mathbb{R}$ if for every $F \Subset \Gamma$ and $p \in A^F$ then $\mu -$ ae

$$\mathbb{E}_\mu(1_{[p]} \mid \sigma(A^{\Gamma \setminus F}))(x) = \begin{cases} \frac{\exp(\Psi_f(x, p \vee x|_{\Gamma \setminus F}))}{\sum_{q \in L_F(X)} \exp(\Psi_f(x, q \vee x|_{\Gamma \setminus F}))} & \text{if } p \in L_F(x) \\ 0 & \text{if } p \notin L_F(x) \end{cases}.$$

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$\mu([p]) = \frac{1}{|A^F|}$ is the uniform Bernoulli measure.

Equilibrium measure

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$$h_\mu(\mathbb{Z}^d \curvearrowright X) = \lim_{n\to\infty} \frac{1}{(2n+1)^d} \sum_{p \in L_{[-n,n]^d}(X)} -\mu([p]) \log(\mu([p])).$$

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Equilibrium measure on $\mathbb{Z}^d$

An invariant Borel probability measure on a $\mathbb{Z}^d$-subshift is of equilibrium if it maximizes

$$h_\mu(\Gamma \curvearrowright X) + \int f \mathrm{d}\mu.$$

Among all invariant Borel probability measures on X.

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For $A^{\mathbb{Z}^d}$ and $f = 0$, there is only one equilibrium and Gibbs measure and they coincide.

Equilibrium measure

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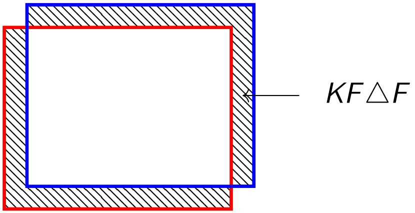
$$K = \{(1,1)\}$$
$$F = [\![-n, n]\!]^2$$



Equilibrium measure

Amenable group

A group Γ is **amenable** if for every $K \Subset \Gamma$ and $\delta > 0$ there is $F \Subset \Gamma$ which is (K, δ)-invariant.

A sequence $(F_n)_{n \in \mathbb{N}}$ of finite subsets of Γ is **Følner** if it is eventually (K, δ)-invariant for every $K \Subset \Gamma$ and $\delta > 0$.

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$$h_\mu(\Gamma \curvearrowright X) = \lim_{n\to\infty} \frac{1}{|F_n|} \sum_{p \in L_{F_n}(X)} -\mu([p]) \log(\mu([p])).$$

Where $(F_n)_{n\in\mathbb{N}}$ is any Følner sequence.

Remark: the limit does not depend on the Følner sequence.

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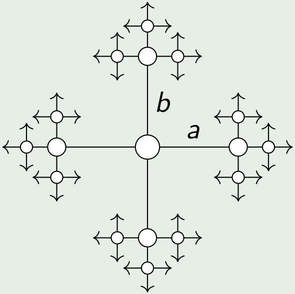

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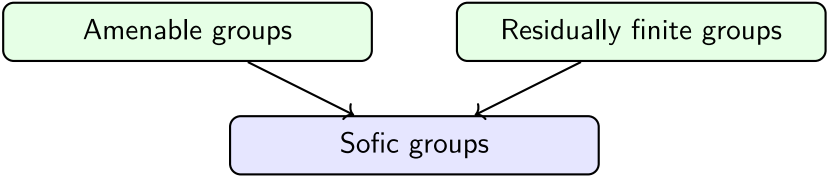
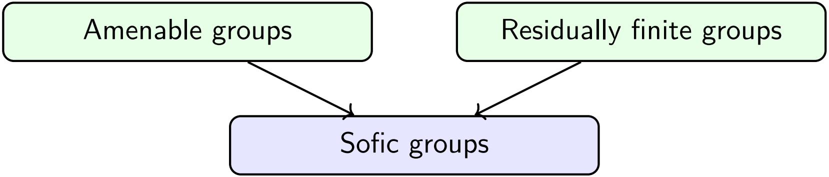
Let $\varphi\colon (\mathbb{Z}/2\mathbb{Z})^{F_2} \to (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})^{F_2}$ be given by

$$\varphi(x)(g) = (x(g) + x(ga), x(g) + x(gb)).$$

φ is a factor map from $(\mathbb{Z}/2\mathbb{Z})^{F_2}$ to $((\mathbb{Z}/2\mathbb{Z})^2)^{F_2}$. One of the two properties must fail on F_2.


Amenable groups
Residually finite groups
Sofic groups



A group Γ is **sofic** if there's a sequence $(V_i)_{i\in\mathbb{N}}$ of finite sets $|V_i| \to \infty$ and a collection $\Sigma = \{\sigma_i\colon \Gamma \to \mathrm{Sym}(V_i)\}_{i\in\mathbb{N}}$ which is:

- Asymptotically an action: For every $s, t \in \Gamma$,

$$\lim_{i\to\infty} \frac{1}{|V_i|} |\{v \in V_i : \sigma_i(st)v = \sigma_i(s)\sigma_i(t)v\}| = 1.$$

- Asymptotically free: For every $s \neq t \in \Gamma$,

$$\lim_{i\to\infty} \frac{1}{|V_i|} |\{v \in V_i : \sigma_i(s)v \neq \sigma_i(t)v\}| = 1.$$

Example: If Γ is residually finite, there is a sequence of normal subgroups $(H_n)_{n\in\mathbb{N}}$ such that $[\Gamma : H_n] < \infty$ and $\bigcap_{n\in\mathbb{N}} H_n = 1$.

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Residually finite groups are sofic.

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$$\sigma_n(g)(f) = \begin{cases} gf & \text{if } gf \in F_n \\ \tau_g(f) & \text{otherwise.} \end{cases}$$

where $\tau_g \colon F_n \setminus g^{-1}F_n$ is some permutation.

Example: If Γ is amenable, consider a Følner sequence $(F_n)_{n\in\mathbb{N}}$ such that $1_\Gamma \in F_n \nearrow \Gamma$.

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The collection $\Sigma = \{\sigma_n \colon \Gamma \to \mathrm{Sym}(F_n)\}_{n\in\mathbb{N}}$ is

- Asymptotically an action. For every $g, h \in \Gamma$ let $K = \{g, h, gh\}$ and $\varepsilon > 0$. There is F_n such that $|KF\Delta F| < \varepsilon|F|$. Therefore the number of $f \in F$ such that $\sigma_n(g)\sigma_n(h)f = \sigma_n(gh)f$ is at least $(1-\varepsilon)|F|$.
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Amenable groups are sofic.

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Fix a sofic approximation sequence $\Sigma = \{\sigma_i \colon \Gamma \to \mathrm{Sym}(V_i)\}_{i\in\mathbb{N}}$ of Γ and a generating pseudometric ρ on X.

$$h_\Sigma(\Gamma \curvearrowright X, \mu) = \sup_{\varepsilon>0} \inf_{L\in C(X)} \inf_{F\in\Gamma} \inf_{\delta>0} \limsup_{i\to\infty} \frac{1}{|V_i|} \log(M^{\varepsilon}_{\Sigma,\mu}(\Gamma \curvearrowright X, F, \delta, L, \sigma_i))$$

Where $M^{\varepsilon}_{\Sigma,\mu}(\Gamma \curvearrowright X, F, \delta, L, \sigma_i)$ is the maximum cardinality of a collection of maps $\varphi \colon V_n \to X$ such that

1. are ε-separated, $\max_{v\in V_i} \rho(\varphi(v), \varphi'(v)) > \varepsilon$.
2. son (F, δ)-close to an orbit

$$\max_{s\in F} \frac{1}{|V_i|} \left(\sum_{v\in V_i} \rho(s\varphi(v), \varphi(\sigma_i(s)v))^2 \right)^{\frac{1}{2}} < \delta.$$

3. Are almost generic with respect to the measure

$$\left| \frac{1}{|V_i|} \sum_{v\in V_i} h(\varphi(v)) - \int_X h \mathrm{d}\mu \right| \leq \delta, \text{ for every } h \in L.$$

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For $w \in A^{V_i}$, define $\varphi_w(v) \in A^\Gamma$ by

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Each $w \in A^{V_i}$ induces a probability measure on A^Γ given by

$$\mu_w = \frac{1}{|V_i|} \sum_{v \in V_i} \delta_{\varphi_w(v)}.$$

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The sofic entropy of a subshift (X, μ) with respect to a sofic approximation sequence Σ is given by:

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An invariant Borel probability measure μ is an **equilibrium measure** for a subshift $X \subset A^\Gamma$ with respect to Σ and $f \colon X \to \mathbb{R}$ if it maximizes the expression:

$$h_\Sigma(\Gamma \curvearrowright X, \mu) + \int f \, \mathrm{d}\mu.$$

Theorem: B., Meyerovitch, 2021

Let Γ be a sofic group and Σ a sofic approximation sequence of Γ. Let $f: X \to \mathbb{R}$ be sufficiently regular. For every Γ-subshift which has TMP and $h_\Sigma(\Gamma \curvearrowright X) \geq 0$, every equilibrium measure μ is Gibbs.

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$h_\Sigma(\Gamma \curvearrowright X) \geq 0$ if and only if there is an invariant Borel probability measure μ such that $h_\Sigma(\Gamma \curvearrowright X, \mu) \geq 0$.

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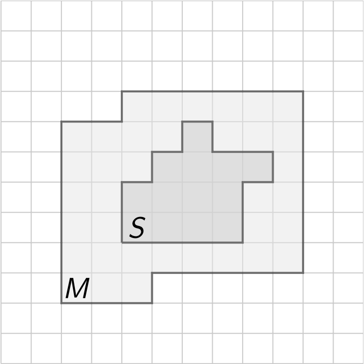

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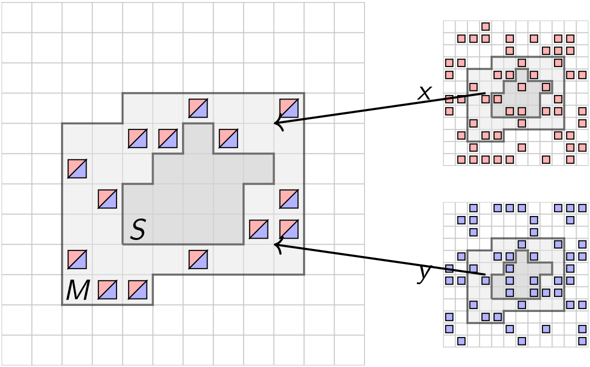

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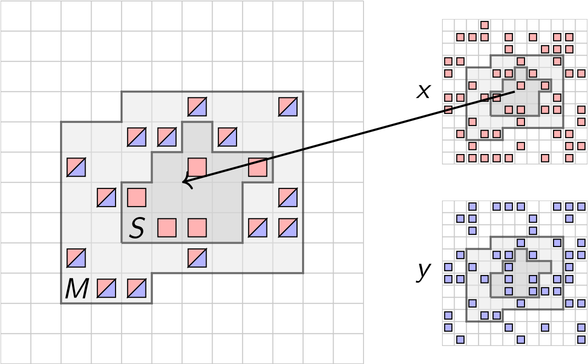

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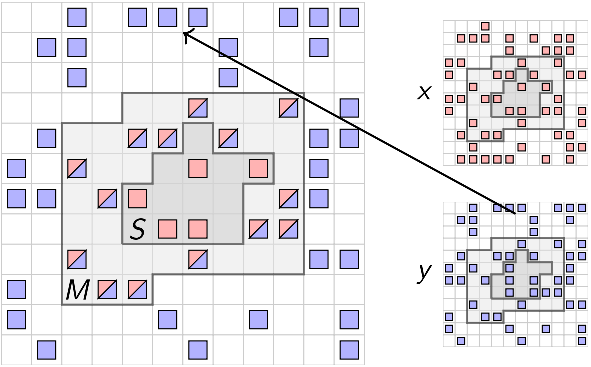

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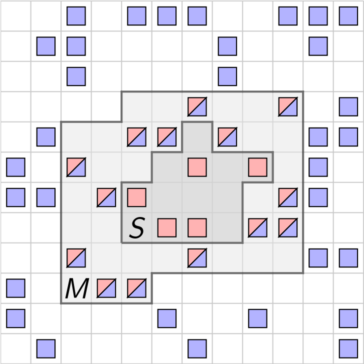

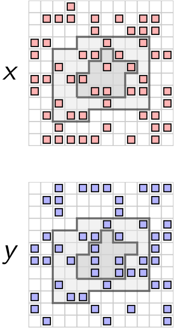

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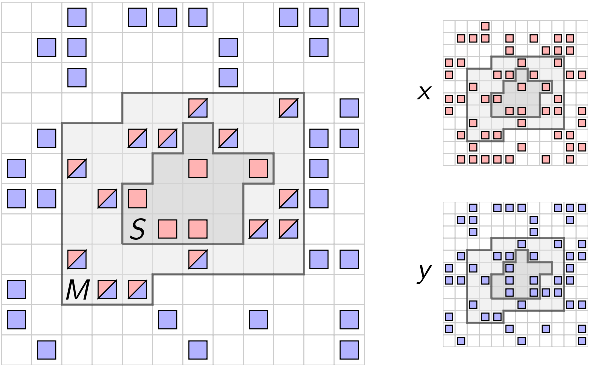

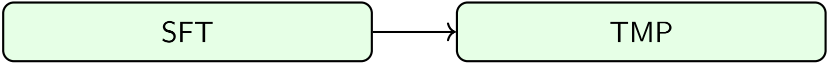
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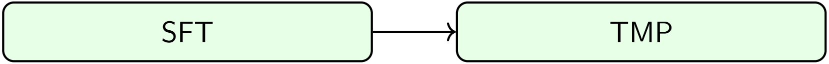
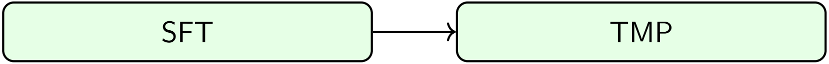


Two patterns $p, q \in A^F$ are **exchangeable** in a subshift $X \subset A^\Gamma$ if for every $x \in X$

$$x|_{\Gamma \setminus F} \vee p \in X \text{ if and only if } x|_{\Gamma \setminus F} \vee q \in X.$$



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Remark: If X has TMP and $M \supset F$ is a memory set for $F \Subset \Gamma$, then for every x, y such that $x|_{\Gamma \setminus F} = y|_{\Gamma \setminus F}$ we have that $x|_M, y|_M$ are exchangeable.

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Theorem

$\mathcal{T}(X) = \mathcal{T}^0(X)$ if and only if X has the TMP.

A Borel probability measure μ on a subshift X is **Gibbs** with respect to $f\colon X \to \mathbb{R}$ if for every $F \Subset \Gamma$ y $p \in A^F$ we have that $\mu -$ ae

$$\mathbb{E}_\mu(1_{[p]} \mid \sigma(A^{\Gamma\setminus F}))(x) = \begin{cases} \frac{\exp(\Psi_f(x, p \vee x|_{\Gamma\setminus F}))}{\sum_{q\in L_F(X)} \exp(\Psi_f(x, q \vee x|_{\Gamma\setminus F}))} & \text{if } p \in L_F(x) \\ 0 & \text{if } p \notin L_F(x) \end{cases}.$$

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For $p, q \in L_F(x)$

$$\frac{\mathbb{E}_{\mu}(1_{[p]} \mid \sigma(A^{\Gamma\setminus F}))(x)}{\mathbb{E}_{\mu}(1_{[q]} \mid \sigma(A^{\Gamma\setminus F}))(x)} = \exp(\Psi_f(q \vee x|_{\Gamma\setminus F}, p \vee x|_{\Gamma\setminus F})).$$

Therefore the log of the above expression forms a cocycle on $\mathcal{T}(X)$.

Let μ be non-singular with respect to a countable Borel equivalence relation $\mathcal{R}$.

$$\left(\text{if } \mu(A) = 0 \implies \mu\left(\bigcup_{x \in A} \{y \in X : (x, y) \in \mathcal{R}\} \right) = 0 \right).$$

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There is a countable group G which generates $\mathcal{R}$ and a map $\mathcal{D}_{\mu,\mathcal{R}} \colon \mathcal{R} \to \mathbb{R}_+$ such that for every $\phi \in G$,

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Let $f \colon X \to \mathbb{R}$ such that the cocycle Ψ_f is well-defined. A Borel probability measure μ on a subshift X is:

1. **Gibbs** with respect to f if it is non-singular with respect to $\mathcal{T}(X)$ and $\mathcal{D}_{\mu,\mathcal{T}(X)} = \exp(\Psi_f)$ μ-ae.
2. **étale Gibbs** with respect to f if it is non-singular with respect to $\mathcal{T}^0(X)$ and $\mathcal{D}_{\mu,\mathcal{T}^0(X)} = \exp(\Psi_f)$ μ-ae.

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- δ_0 is étale Gibbs because $\mathcal{T}^0($$) = \{(x,x) : x \in$ $\} \cup (Y \times Y)$ for $Y = \{x \in \{0,1\}^{\mathbb{Z}} : |x^{-1}(1)| = 1\}$ and the measure is supported on  $\setminus Y$.

Proof sketch

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2. Let $x, y \in \mathcal{T}^0(X)$ and $F \Subset \Gamma$ such that $p = x|_F$ and $q = y|_F$ do not **overlap**. Consider the set $\mathcal{U}(t, r, \delta)$ of measures μ s.t.

$$|\mu([p] \cup [q]) - t| < \delta \text{ and } \left|\frac{\mu([p])}{\mu([q])} - r\right| < \delta.$$

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   We show that for $r, s \geq 0$, if we let $r^* = \frac{r}{1+r}$ and $s^* = \frac{s}{1+s}$ then the difference on the expressions for sofic pressure $P(\mathcal{U}(t, r, \delta)) - P(\mathcal{U}(t, s, \delta))$ can be parameterized as

$$t\left(H(r^*) + r^*\Psi_f(x, y) - H(s^*) - s^*\Psi_f(x, y)\right).$$

3. If $t > 0$, the maximum value is obtained at $r = \exp(\Psi_f(x, y))$.

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2. The rest of the proof consists on getting rid of the technical condition (taking a product with a Bernoulli shift and showing that the non-overlap condition is generic).
3. The generalization to non-local functions f is done through functional analysis by interpreting equilibrium measures as subderivatives (and using a Mazur-like theorem).

Functions that work with LR

A translation-invariant map $\Phi\colon L(X) \to \mathbb{R}$ is called an **interaction**, we say Φ is **absolutely summable** if

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Let $f_\Phi\colon X \to \mathbb{R}$ be given by

$$f_\Phi(x) = \sum_{1_G \in F \Subset \Gamma} \frac{1}{|F|} \Phi(x|_F).$$

Maps f_Φ for some absolutely summable Φ satisfy LR's theorem.

Functions that work with LR

Let $\mathbb{F} = (F_n)_{n\in\mathbb{N}}$ be a sequence of finite subsets of Γ such that $F_n \nearrow \Gamma$. Given $f\colon X \to \mathbb{R}$ and $S \Subset \Gamma$

$$\mathrm{Var}_S(f) = \sup\{|f(x) - f(y)| : x, y \in X, x|_S = y|_S\}.$$

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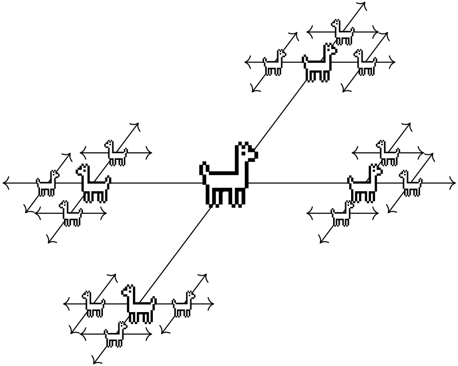
3. Suppose A is a finite group and $X \subset A^\Gamma$ is a subshift which is closed under pointwise multiplication. If the homoclinic group

   $$\Delta(X) = \{x \in X : (x, e) \in \mathcal{T}(X)\}$$

   is dense in X, then the Haar measure is the unique measure of maximal entropy for every sofic approximation sequence.

Thanks!



The Lanford–Ruelle theorem for actions of sofic groups
S. Barbieri, T. Meyerovitch
Transactions of the AMS, to appear.
https://arxiv.org/abs/2112.02334