Insdistinguishability and multidimensional Sturmian configurations

Sebastián **Barbieri Lemp**
On joint work with S. Labbé and Š. Starosta

Universidad de Santiago de Chile

Dyadisc 5
June, 2022

- Let A be a finite set and $d \geq 1$ be an integer.
- A configuration is a map $x \colon \mathbb{Z}^d \to A$.

- Let A be a finite set and $d \geq 1$ be an integer.
- A configuration is a map $x \colon \mathbb{Z}^d \to A$.

For example if $d = 2$ and $A = \{0, 1, 2\}$ a configuration looks like:

1	0	2	2	1	0	2	1	0	2	1	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0	2	2	1	0
2	1	0	2	2	1	0	2	1	0	2	1	1	0	2
1	0	2	1	1	0	2	1	0	2	1	0	2	2	1
0	2	1	0	2	2	1	0	2	1	0	2	1	1	0
2	1	0	2	1	1	0	2	1	0	2	1	0	2	2
1	0	2	1	0	2	2	1	0	2	1	0	2	1	1
0	2	1	0	2	1	1	0	2	1	0	2	1	0	2
2	1	0	2	1	0	2	2	1	0	2	1	0	2	1
1	0	2	1	0	2	1	0	2	2	1	0	2	1	0
2	2	1	0	2	1	0	2	1	1	0	2	1	0	2
1	1	0	2	1	0	2	1	0	2	2	1	0	2	1
0	2	2	1	0	2	1	0	2	1	1	0	2	1	0
2	1	1	0	2	1	0	2	1	0	2	2	1	0	2
1	0	2	2	1	0	2	1	0	2	1	1	0	2	1

We say two configurations $x, y \in A^{\mathbb{Z}^d}$ are **asymptotic** if there exists a finite $F \subset \mathbb{Z}^d$ such that $x|_{\mathbb{Z}^d \setminus F} = y|_{\mathbb{Z}^d \setminus F}$.

We say two configurations $x, y \in A^{\mathbb{Z}^d}$ are **asymptotic** if there exists a finite $F \subset \mathbb{Z}^d$ such that $x|_{\mathbb{Z}^d \setminus F} = y|_{\mathbb{Z}^d \setminus F}$.

Given asymptotic x, y, we call $F = \{n \in \mathbb{Z}^d : x_n \neq y_n\}$ their **difference set**.

x

1	0	2	2	1	0	2	1	0	2	1	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0	2	2	1	0
2	1	0	2	2	1	0	2	1	0	2	1	1	0	2
1	0	2	1	1	0	2	1	0	2	1	0	2	2	1
0	2	1	0	2	2	1	0	2	1	0	2	1	1	0
2	1	0	2	1	1	0	2	1	0	2	1	0	2	2
1	0	2	1	0	2	2	1	0	2	1	0	2	1	1
0	2	1	0	2	1	1	0	2	1	0	2	1	0	2
2	1	0	2	1	0	2	2	1	0	2	1	0	2	1
1	0	2	1	0	2	1	0	2	2	1	0	2	1	0
2	2	1	0	2	1	0	2	1	1	0	2	1	0	2
1	1	0	2	1	0	2	1	0	2	2	1	0	2	1
0	2	2	1	0	2	1	0	2	1	1	0	2	1	0
2	1	1	0	2	1	0	2	1	0	2	2	1	0	2
1	0	2	2	1	0	2	1	0	2	1	1	0	2	1

y

1	0	2	2	1	0	2	1	0	2	1	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0	2	2	1	0
2	1	0	2	2	1	0	2	1	0	2	1	1	0	2
1	0	2	1	1	0	2	1	0	2	1	0	2	2	1
0	2	1	0	2	2	1	0	2	1	0	2	1	1	0
2	1	0	2	1	1	0	2	1	0	2	1	0	2	2
1	0	2	1	0	2	2	1	0	2	1	0	2	1	1
0	2	1	0	2	1	0	2	2	1	0	2	1	0	2
2	1	0	2	1	0	2	1	1	0	2	1	0	2	1
1	0	2	1	0	2	1	0	2	2	1	0	2	1	0
2	2	1	0	2	1	0	2	1	1	0	2	1	0	2
1	1	0	2	1	0	2	1	0	2	2	1	0	2	1
0	2	2	1	0	2	1	0	2	1	1	0	2	1	0
2	1	1	0	2	1	0	2	1	0	2	2	1	0	2
1	0	2	2	1	0	2	1	0	2	1	1	0	2	1

Why the name “asymptotic”?

Why the name "asymptotic"?

Let σ be the $\mathbb{Z}^d$ action of $A^{\mathbb{Z}^d}$ given by

$$\sigma^n(x)(m) = x(n+m) \text{ for every } n, m \in \mathbb{Z}^d.$$

x, y are asymptotic if and only if for any sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{Z}^d$ with $\|n_k\| \to \infty$ then $d(\sigma^{n_k}(x), \sigma^{n_k}(y)) \to 0$.

Why the name "asymptotic"?

Let σ be the $\mathbb{Z}^d$ action of $A^{\mathbb{Z}^d}$ given by

$$\sigma^n(x)(m) = x(n+m) \text{ for every } n, m \in \mathbb{Z}^d.$$

x, y are asymptotic if and only if for any sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{Z}^d$ with $\|n_k\| \to \infty$ then $d(\sigma^{n_k}(x), \sigma^{n_k}(y)) \to 0$.

$x =$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$y =$	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0

Why the name "asymptotic"?

Let σ be the $\mathbb{Z}^d$ action of $A^{\mathbb{Z}^d}$ given by

$$\sigma^n(x)(m) = x(n+m) \text{ for every } n, m \in \mathbb{Z}^d.$$

x, y are asymptotic if and only if for any sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{Z}^d$ with $\|n_k\| \to \infty$ then $d(\sigma^{n_k}(x), \sigma^{n_k}(y)) \to 0$.

$\sigma(x) =$

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$\sigma(y) =$

0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0

Why the name "asymptotic"?

Let σ be the $\mathbb{Z}^d$ action of $A^{\mathbb{Z}^d}$ given by

$$\sigma^n(x)(m) = x(n+m) \text{ for every } n, m \in \mathbb{Z}^d.$$

> x, y are asymptotic if and only if for any sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{Z}^d$ with $\|n_k\| \to \infty$ then $d(\sigma^{n_k}(x), \sigma^{n_k}(y)) \to 0$.

$\sigma^2(x) =$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\sigma^2(y) =$	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0

Why the name "asymptotic"?

Let σ be the $\mathbb{Z}^d$ action of $A^{\mathbb{Z}^d}$ given by

$$\sigma^n(x)(m) = x(n+m) \text{ for every } n, m \in \mathbb{Z}^d.$$

x, y are asymptotic if and only if for any sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{Z}^d$ with $\|n_k\| \to \infty$ then $d(\sigma^{n_k}(x), \sigma^{n_k}(y)) \to 0$.

$\sigma^3(x) =$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\sigma^3(y) =$	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0

Why the name "asymptotic"?

Let σ be the $\mathbb{Z}^d$ action of $A^{\mathbb{Z}^d}$ given by

$$\sigma^n(x)(m) = x(n+m) \text{ for every } n, m \in \mathbb{Z}^d.$$

x, y are asymptotic if and only if for any sequence $(n_k)_{k \in \mathbb{N}}$ in $\mathbb{Z}^d$ with $\|n_k\| \to \infty$ then $d(\sigma^{n_k}(x), \sigma^{n_k}(y)) \to 0$.

$\sigma^4(x) =$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\sigma^4(y) =$	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Why the name "asymptotic"?

Let σ be the $\mathbb{Z}^d$ action of $A^{\mathbb{Z}^d}$ given by

$$\sigma^n(x)(m) = x(n+m) \text{ for every } n, m \in \mathbb{Z}^d.$$

> x, y are asymptotic if and only if for any sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{Z}^d$ with $\|n_k\| \to \infty$ then $d(\sigma^{n_k}(x), \sigma^{n_k}(y)) \to 0$.

$\sigma^5(x) =$ 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

$\sigma^5(y) =$ 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

Why the name "asymptotic"?

Let σ be the $\mathbb{Z}^d$ action of $A^{\mathbb{Z}^d}$ given by

$$\sigma^n(x)(m) = x(n+m) \text{ for every } n, m \in \mathbb{Z}^d.$$

x, y are asymptotic if and only if for any sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{Z}^d$ with $\|n_k\| \to \infty$ then $d(\sigma^{n_k}(x), \sigma^{n_k}(y)) \to 0$.

$\sigma^6(x) =$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\sigma^6(y) =$	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Why the name “asymptotic”?

Let σ be the $\mathbb{Z}^d$ action of $A^{\mathbb{Z}^d}$ given by

$$\sigma^n(x)(m) = x(n+m) \text{ for every } n, m \in \mathbb{Z}^d.$$

x, y are asymptotic if and only if for any sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{Z}^d$ with $\|n_k\| \to \infty$ then $d(\sigma^{n_k}(x), \sigma^{n_k}(y)) \to 0$.

$\sigma^7(x) =$

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$\sigma^7(y) =$

0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Why the name "asymptotic"?

Let σ be the $\mathbb{Z}^d$ action of $A^{\mathbb{Z}^d}$ given by

$$\sigma^n(x)(m) = x(n+m) \text{ for every } n, m \in \mathbb{Z}^d.$$

x, y are asymptotic if and only if for any sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{Z}^d$ with $\|n_k\| \to \infty$ then $d(\sigma^{n_k}(x), \sigma^{n_k}(y)) \to 0$.

$\sigma^9(x) =$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\sigma^9(y) =$	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Why the name "asymptotic"?

Let σ be the $\mathbb{Z}^d$ action of $A^{\mathbb{Z}^d}$ given by

$$\sigma^n(x)(m) = x(n+m) \text{ for every } n, m \in \mathbb{Z}^d.$$

> x, y are asymptotic if and only if for any sequence $(n_k)_{k \in \mathbb{N}}$ in $\mathbb{Z}^d$ with $\|n_k\| \to \infty$ then $d(\sigma^{n_k}(x), \sigma^{n_k}(y)) \to 0$.

$\sigma^{10}(x) =$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\sigma^{10}(y) =$	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Why the name "asymptotic"?

Let σ be the $\mathbb{Z}^d$ action of $A^{\mathbb{Z}^d}$ given by

$$\sigma^n(x)(m) = x(n+m) \text{ for every } n, m \in \mathbb{Z}^d.$$

x, y are asymptotic if and only if for any sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{Z}^d$ with $\|n_k\| \to \infty$ then $d(\sigma^{n_k}(x), \sigma^{n_k}(y)) \to 0$.

$\sigma^{11}(x) =$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\sigma^{11}(y) =$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

- Let $x, y \in A^{\mathbb{Z}^d}$ be asymptotic.
- Given $S \Subset \mathbb{Z}^d$ and a pattern $p \in A^S$ let

$$[p] = \{z \in A^{\mathbb{Z}^d} : z|_S = p\}.$$

- Let $x, y \in A^{\mathbb{Z}^d}$ be asymptotic.
- Given $S \Subset \mathbb{Z}^d$ and a pattern $p \in A^S$ let

$$[p] = \{z \in A^{\mathbb{Z}^d} : z|_S = p\}.$$

We wish to compute how many times p occurs in x vs how many times it occurs in y.

- Let $x, y \in A^{\mathbb{Z}^d}$ be asymptotic.
- Given $S \Subset \mathbb{Z}^d$ and a pattern $p \in A^S$ let

$$[p] = \{z \in A^{\mathbb{Z}^d} : z|_S = p\}.$$

We wish to compute how many times p occurs in x vs how many times it occurs in y.

$$\Delta_p(x, y) = \sum_{u \in \mathbb{Z}^d} 1_{[p]}(\sigma^u(y)) - 1_{[p]}(\sigma^u(x)).$$

- Let $x, y \in A^{\mathbb{Z}^d}$ be asymptotic.
- Given $S \Subset \mathbb{Z}^d$ and a pattern $p \in A^S$ let

$$[p] = \{z \in A^{\mathbb{Z}^d} : z|_S = p\}.$$

We wish to compute how many times p occurs in x vs how many times it occurs in y.

$$\Delta_p(x, y) = \sum_{u \in \boxed{F - S}} 1_{[p]}(\sigma^u(y)) - 1_{[p]}(\sigma^u(x)).$$

- Let $x, y \in A^{\mathbb{Z}^d}$ be asymptotic.
- Given $S \Subset \mathbb{Z}^d$ and a pattern $p \in A^S$ let

$$[p] = \{z \in A^{\mathbb{Z}^d} : z|_S = p\}.$$

We wish to compute how many times p occurs in x vs how many times it occurs in y.

$$\Delta_p(x, y) = \sum_{u \in \boxed{F - S}} 1_{[p]}(\sigma^u(y)) - 1_{[p]}(\sigma^u(x)).$$

We say an asymptotic pair x, y is indistinguishable if $\Delta_p(x, y) = 0$ for every pattern p.

Example: Let $d = 2$ and $S = \{(0,0),(0,1),(1,0),(2,0)\}$.

x

1	0	2	2	1	0	2	1	0	2	1	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0	2	2	1	0
2	1	0	2	2	1	0	2	1	0	2	1	1	0	2
1	0	2	1	1	0	2	1	0	2	1	0	2	2	1
0	2	1	0	2	2	1	0	2	1	0	2	1	1	0
2	1	0	2	1	1	0	2	1	0	2	1	0	2	2
1	0	2	1	0	2	2	1	0	2	1	0	2	1	1
0	2	1	0	2	1	1	0	2	1	0	2	1	0	2
2	1	0	2	1	0	2	2	1	0	2	1	0	2	1
1	0	2	1	0	2	1	0	2	2	1	0	2	1	0
2	2	1	0	2	1	0	2	1	1	0	2	1	0	2
1	1	0	2	1	0	2	1	0	2	2	1	0	2	1
0	2	2	1	0	2	1	0	2	1	1	0	2	1	0
2	1	1	0	2	1	0	2	1	0	2	2	1	0	2
1	0	2	2	1	0	2	1	0	2	1	1	0	2	1

y

1	0	2	2	1	0	2	1	0	2	1	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0	2	2	1	0
2	1	0	2	2	1	0	2	1	0	2	1	1	0	2
1	0	2	1	1	0	2	1	0	2	1	0	2	2	1
0	2	1	0	2	2	1	0	2	1	0	2	1	1	0
2	1	0	2	1	1	0	2	1	0	2	1	0	2	2
1	0	2	1	0	2	2	1	0	2	1	0	2	1	1
0	2	1	0	2	1	0	2	2	1	0	2	1	0	2
2	1	0	2	1	0	2	1	1	0	2	1	0	2	1
1	0	2	1	0	2	1	0	2	2	1	0	2	1	0
2	2	1	0	2	1	0	2	1	1	0	2	1	0	2
1	1	0	2	1	0	2	1	0	2	2	1	0	2	1
0	2	2	1	0	2	1	0	2	1	1	0	2	1	0
2	1	1	0	2	1	0	2	1	0	2	2	1	0	2
1	0	2	2	1	0	2	1	0	2	1	1	0	2	1

Example: Let $d = 2$ and $S = \{(0,0),(0,1),(1,0),(2,0)\}$.

x

0	2	2	1	0	2
2	1	1	0	2	1
1	0	2	2	1	0
0	2	1	0	2	2

y

0	2	2	1	0	2
2	1	0	2	2	1
1	0	2	1	1	0
0	2	1	0	2	2

Example: Let $d = 2$ and $S = \{(0,0),(0,1),(1,0),(2,0)\}$.

x

0	2	2	1	0	2
2	1	1	0	2	1
1	0	2	2	1	0
0	2	1	0	2	2

y

0	2	2	1	0	2
2	1	0	2	2	1
1	0	2	1	1	0
0	2	1	0	2	2

Example: Let $d = 2$ and $S = \{(0,0),(0,1),(1,0),(2,0)\}$.

x

0	2	2	1	0	2
2	1	1	0	2	1
1	0	2	2	1	0
0	2	1	0	2	2

y

0	2	2	1	0	2
2	1	0	2	2	1
1	0	2	1	1	0
0	2	1	0	2	2

Example: Let $d = 2$ and $S = \{(0,0),(0,1),(1,0),(2,0)\}$.

x

0	2	2	1	0	2
2	1	1	0	2	1
1	0	2	2	1	0
0	2	1	0	2	2

y

0	2	2	1	0	2
2	1	0	2	2	1
1	0	2	1	1	0
0	2	1	0	2	2

Example: Let $d = 2$ and $S = \{(0,0),(0,1),(1,0),(2,0)\}$.

x

0	2	2	1	0	2
2	1	1	0	2	1
1	0	2	2	1	0
0	2	1	0	2	2

y

0	2	2	1	0	2
2	1	0	2	2	1
1	0	2	1	1	0
0	2	1	0	2	2

Example: Let $d = 2$ and $S = \{(0,0),(0,1),(1,0),(2,0)\}$.

x

0	2	2	1	0	2
2	1	1	0	2	1
1	0	2	2	1	0
0	2	1	0	2	2

y

0	2	2	1	0	2
2	1	0	2	2	1
1	0	2	1	1	0
0	2	1	0	2	2

Example: Let $d = 2$ and $S = \{(0,0),(0,1),(1,0),(2,0)\}$.

x

0	2	2	1	0	2
2	1	1	0	2	1
1	0	2	2	1	0
0	2	1	0	2	2

y

0	2	2	1	0	2
2	1	0	2	2	1
1	0	2	1	1	0
0	2	1	0	2	2

Example: Let $d = 2$ and $S = \{(0,0),(0,1),(1,0),(2,0)\}$.

x

0	2	2	1	0	2
2	1	1	0	2	1
1	0	2	2	1	0
0	2	1	0	2	2

y

0	2	2	1	0	2
2	1	0	2	2	1
1	0	2	1	1	0
0	2	1	0	2	2

Example: Let $d = 2$ and $S = \{(0,0),(0,1),(1,0),(2,0)\}$.

x

0	2	2	1	0	2
2	1	1	0	2	1
1	0	2	2	1	0
0	2	1	0	2	2

y

0	2	2	1	0	2
2	1	0	2	2	1
1	0	2	1	1	0
0	2	1	0	2	2

Example: Let $d = 2$ and $S = \{(0,0),(0,1),(1,0),(2,0)\}$.

x

0	2	2	1	0	2
2	1	1	0	2	1
1	0	2	2	1	0
0	2	1	0	2	2

y

0	2	2	1	0	2
2	1	0	2	2	1
1	0	2	1	1	0
0	2	1	0	2	2

So for every pattern p with support S, we have $\Delta_p(x, y) = 0$.

Examples:

- (x, x) for any $x \in A^{\mathbb{Z}^d}$ is an indistinguishable asymptotic pair. We call it **trivial**.

Examples:

- (x, x) for any $x \in A^{\mathbb{Z}^d}$ is an indistinguishable asymptotic pair. We call it **trivial**.
- If $x, y \in A^{\mathbb{Z}^d}$ are asymptotic and on the same orbit ($\sigma^n(y) = x$ for some $n \in \mathbb{Z}^d$) then they are indistinguishable.

x

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	0	1	0	0
0	0	0	0	1	1	1	1	1	0
0	0	0	0	1	1	1	1	1	0
0	0	0	0	0	1	1	1	0	0
0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

y

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	1	0	1	0	0	0	0	0
0	1	1	1	1	1	0	0	0	0
0	1	1	1	1	1	0	0	0	0
0	0	1	1	1	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

Does there exist indistinguishable asymptotic pairs which are not on the same orbit?

Motivation

Consider n balls with real weights given by a map f.

1	2	3	4	5	6	7	$\cdots$	n
$f(1)$	$f(2)$	$f(3)$	$f(4)$	$f(5)$	$f(6)$	$f(7)$		$f(n)$

Motivation

Consider n balls with real weights given by a map f.

1	2	3	4	5	6	7	$\cdots$	n
$f(1)$	$f(2)$	$f(3)$	$f(4)$	$f(5)$	$f(6)$	$f(7)$		$f(n)$

¿What is the probability distribution $\mu = (\mu_1, \ldots, \mu_n)$ on $\{1, \ldots, n\}$ that maximizes entropy plus average weight?

$$\max_{\mu} \left(H(\mu) + \int f \,\mathrm{d}\mu \right) = \max_{\mu} \sum_{i=1}^{n} \left(-\mu_i \log(\mu_i) + f(i)\mu_i \right).$$

Motivation

Consider n balls with real weights given by a map f.

1	2	3	4	5	6	7	$\cdots$	n
$f(1)$	$f(2)$	$f(3)$	$f(4)$	$f(5)$	$f(6)$	$f(7)$		$f(n)$

¿What is the probability distribution $\mu = (\mu_1, \ldots, \mu_n)$ on $\{1, \ldots, n\}$ that maximizes entropy plus average weight?

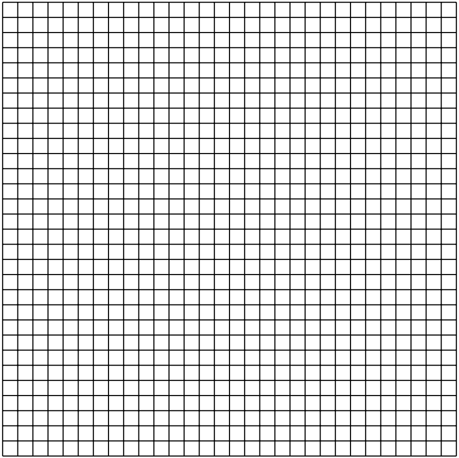
$$\max_{\mu} \left(H(\mu) + \int f \,\mathrm{d}\mu \right) = \max_{\mu} \sum_{i=1}^{n} \left(-\mu_i \log(\mu_i) + f(i)\mu_i \right).$$

Answer: Boltzmann's distribution.

$$\mu_k = \frac{\exp(f(k))}{\sum_{i=1}^{n} \exp(f(i))}.$$

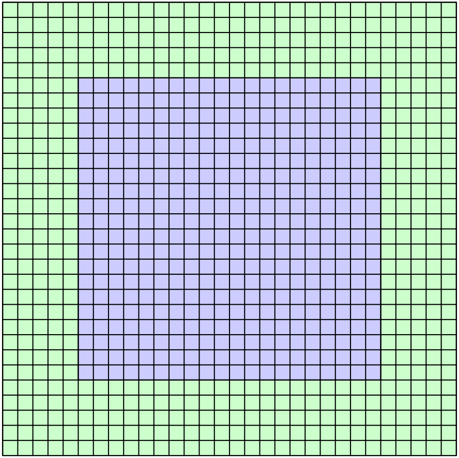
We can extend this idea to sets of configurations, yielding the notion of **Gibbs measures**.

Gibbs measures



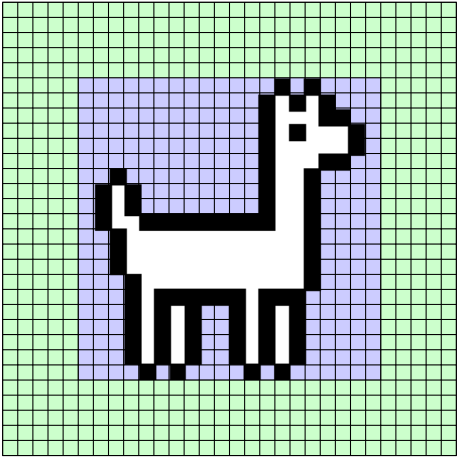
A measure is Gibbs if it follows Boltzmann's distribution on its asymptotic relation.

Gibbs measures



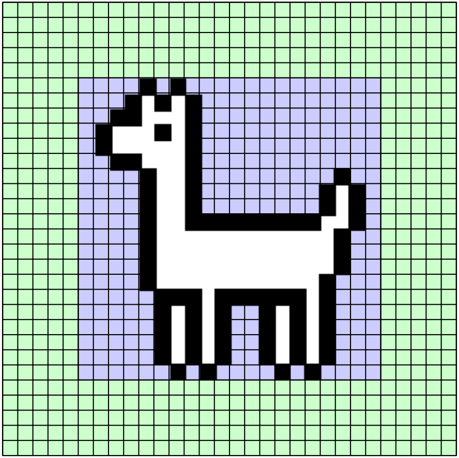
A measure is Gibbs if it follows Boltzmann's distribution on its asymptotic relation.

Gibbs measures



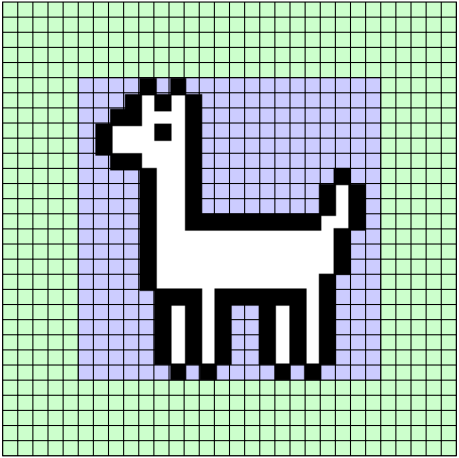
A measure is Gibbs if it follows Boltzmann's distribution on its asymptotic relation.

Gibbs measures



A measure is Gibbs if it follows Boltzmann's distribution on its asymptotic relation.

Gibbs measures



$\mu($$|$$)$ and $\mu($$|$$)$ follow Boltzmann's distribution for some f.

Gibbs Measures

Denote the set of all asymptotic pairs (x, y) by $\mathcal{A}$. The Boltzmann distribution of a Gibbs measure is determined by a **cocycle** $\Psi\colon \mathcal{A} \to \mathbb{R}$, that is, a map which satisfies:

$$\Psi(x, y) = \Psi(x, z) + \Psi(z, y) \text{ for all } (x, y), (y, z) \in \mathcal{A}.$$

Gibbs Measures

Denote the set of all asymptotic pairs (x, y) by $\mathcal{A}$. The Boltzmann distribution of a Gibbs measure is determined by a **cocycle** $\Psi\colon \mathcal{A} \to \mathbb{R}$, that is, a map which satisfies:

$$\Psi(x, y) = \Psi(x, z) + \Psi(z, y) \text{ for all } (x, y), (y, z) \in \mathcal{A}.$$

Example: if all configurations all equally likely (that is, there is no associated weight) we obtain the cocycle $\Psi = 0$ and the sole Gibbs measure for Ψ is the uniform Bernoulli measure on $A^{\mathbb{Z}^d}$.

1. The space of continuous, shift-invariant cocycles $\mathcal{B}$ is a Banach space with an appropriate norm.

1. The space of continuous, shift-invariant cocycles $\mathcal{B}$ is a Banach space with an appropriate norm.
2. There is a natural evaluation map on $\mathcal{B}^*$. For $(x, y) \in \mathcal{A}$ we have $\mathrm{ev}_{x,y} \in \mathcal{B}^*$ given by

$$\mathrm{ev}_{x,y}(\Psi) = \Psi(x, y) \text{ for every } \Psi \in \mathcal{B}.$$

1. The space of continuous, shift-invariant cocycles $\mathcal{B}$ is a Banach space with an appropriate norm.
2. There is a natural evaluation map on $\mathcal{B}^*$. For $(x, y) \in \mathcal{A}$ we have $\mathrm{ev}_{x,y} \in \mathcal{B}^*$ given by

$$\mathrm{ev}_{x,y}(\Psi) = \Psi(x, y) \text{ for every } \Psi \in \mathcal{B}.$$

3. It can be shown that the strong norm on $\mathcal{B}^*$ is given by

$$\|\mathrm{ev}_{x,y}\| = \sup_{S \in \mathbb{Z}^d} \frac{1}{|S|} \sum_{p \in A^S} |\Delta_p(x, y)|.$$

1. The space of continuous, shift-invariant cocycles $\mathcal{B}$ is a Banach space with an appropriate norm.
2. There is a natural evaluation map on $\mathcal{B}^*$. For $(x, y) \in \mathcal{A}$ we have $\mathrm{ev}_{x,y} \in \mathcal{B}^*$ given by

$$\mathrm{ev}_{x,y}(\Psi) = \Psi(x, y) \text{ for every } \Psi \in \mathcal{B}.$$

3. It can be shown that the strong norm on $\mathcal{B}^*$ is given by

$$\|\mathrm{ev}_{x,y}\| = \sup_{S \Subset \mathbb{Z}^d} \frac{1}{|S|} \sum_{p \in A^S} |\Delta_p(x, y)|.$$

4. **An asymptotic pair gives the trivial linear functional if and only if it is indistinguishable.**

Does there exist indistinguishable asymptotic pairs which are not on the same orbit?

Does there exist indistinguishable asymptotic pairs which are not on the same orbit?

TM (2018): Probably not, it should be possible to prove this using the Fine and Wilf theorem.

Does there exist indistinguishable asymptotic pairs which are not on the same orbit?

TM (2018): Probably not, it should be possible to prove this using the Fine and Wilf theorem.
SL (2019): Probably not, it should be possible to prove this using the Fine and Wilf theorem.

Does there exist indistinguishable asymptotic pairs which are not on the same orbit?

TM (2018): Probably not, it should be possible to prove this using the Fine and Wilf theorem.
SL (2019): Probably not, it should be possible to prove this using the Fine and Wilf theorem.
ŠS (2019): Probably not, it should be possible to prove this using the Fine and Wilf theorem.

Does there exist indistinguishable asymptotic pairs which are not on the same orbit?

TM (2018): Probably not, it should be possible to prove this using the Fine and Wilf theorem.
SL (2019): Probably not, it should be possible to prove this using the Fine and Wilf theorem.
ŠS (2019): Probably not, it should be possible to prove this using the Fine and Wilf theorem.

Theorem (SB + SL + ŠS, 2021)

Yes.

Does there exist indistinguishable asymptotic pairs which are not on the same orbit?

TM (2018): Probably not, it should be possible to prove this using the Fine and Wilf theorem.
SL (2019): Probably not, it should be possible to prove this using the Fine and Wilf theorem.
ŠS (2019): Probably not, it should be possible to prove this using the Fine and Wilf theorem.

Theorem (SB + SL + ŠS, 2021)

Yes. We completely characterize them on $\mathbb{Z}$.

Does there exist indistinguishable asymptotic pairs which are not on the same orbit?

TM (2018): Probably not, it should be possible to prove this using the Fine and Wilf theorem.
SL (2019): Probably not, it should be possible to prove this using the Fine and Wilf theorem.
ŠS (2019): Probably not, it should be possible to prove this using the Fine and Wilf theorem.

Theorem (SB + SL + ŠS, 2021)

Yes. We completely characterize them on $\mathbb{Z}$. They are closely connected to Sturmian codings of irrational rotations.

Basic properties of indistinguishable pairs:

(x, y) is indistinguishable if and only if $\Delta_p(x, y) = 0$ for every $S \Subset \mathbb{Z}^d$ and pattern $p \in A^S$.

Basic properties of indistinguishable pairs:

(x, y) is indistinguishable if and only if $\Delta_p(x, y) = 0$ for every $S \Subset \mathbb{Z}^d$ and pattern $p \in A^S$.

Let $(S_n)_{n\in\mathbb{N}}$ with $S_n \nearrow \mathbb{Z}^d$. Then (x, y) is indistinguishable if and only if $\Delta_p(x, y) = 0$ for every pattern p with support some S_n.

In particular, it suffices to check the property on rectangular patterns (or words in the case of $\mathbb{Z}$).

Basic properties of indistinguishable pairs:

(x, y) is indistinguishable if and only if $\Delta_p(x, y) = 0$ for every $S \Subset \mathbb{Z}^d$ and pattern $p \in A^S$.

Let $(S_n)_{n\in\mathbb{N}}$ with $S_n \nearrow \mathbb{Z}^d$. Then (x, y) is indistinguishable if and only if $\Delta_p(x, y) = 0$ for every pattern p with support some S_n.

In particular, it suffices to check the property on rectangular patterns (or words in the case of $\mathbb{Z}$).

Indistinguishable asymptotic pairs are invariant under actions of the affine group of $\mathbb{Z}^d$.

In particular, they are invariant under the shift map.

Basic properties of indistinguishable pairs:

If (x, y) is an indistinguishable asymptotic pair and τ is a sliding block code, then $(\tau(x), \tau(y))$ is an indistinguishable asymptotic pair.

Basic properties of indistinguishable pairs:

If (x, y) is an indistinguishable asymptotic pair and τ is a sliding block code, then $(\tau(x), \tau(y))$ is an indistinguishable asymptotic pair.

We say a sequence of asymptotic pairs $(x_n, y_n)_{n\in\mathbb{N}}$ converges in the asymptotic relation to (x, y) if both $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ converge to x, y respectively, and there is $F \Subset \mathbb{Z}^d$ such that the difference set of (x_n, y_n) is contained in F for every $n \in \mathbb{N}$.

Basic properties of indistinguishable pairs:

If (x, y) is an indistinguishable asymptotic pair and τ is a sliding block code, then $(\tau(x), \tau(y))$ is an indistinguishable asymptotic pair.

We say a sequence of asymptotic pairs $(x_n, y_n)_{n \in \mathbb{N}}$ converges in the asymptotic relation to (x, y) if both $(x_n)_{n \in \mathbb{N}}$, $(y_n)_{n \in \mathbb{N}}$ converge to x, y respectively, and there is $F \Subset \mathbb{Z}^d$ such that the difference set of (x_n, y_n) is contained in F for every $n \in \mathbb{N}$.

If $(x_n, y_n)_{n \in \mathbb{N}}$ converges in the asymptotic relation to (x, y) and every pair (x_n, y_n) is indistinguishable, then (x, y) is indistinguishable.

Basic properties of indistinguishable pairs:

Let (x, y) be a non-trivial indistinguishable asymptotic pair. If x is not recurrent, then x and y lie on the same orbit.

Basic properties of indistinguishable pairs:

Let (x, y) be a non-trivial indistinguishable asymptotic pair. If x is not recurrent, then x and y lie on the same orbit.

Proof: Suppose x is not recurrent. Then there exists $p \in A^S$ which occurs at x exactly once (say $\sigma^k(x) \in [p]$).

Basic properties of indistinguishable pairs:

Let (x, y) be a non-trivial indistinguishable asymptotic pair. If x is not recurrent, then x and y lie on the same orbit.

Proof: Suppose x is not recurrent. Then there exists $p \in A^S$ which occurs at x exactly once (say $\sigma^k(x) \in [p]$).

1. As x, y are indistinguishable, p also occurs exactly once on y, say $\sigma^m(y) \in [p]$.

Basic properties of indistinguishable pairs:

Let (x, y) be a non-trivial indistinguishable asymptotic pair. If x is not recurrent, then x and y lie on the same orbit.

Proof: Suppose x is not recurrent. Then there exists $p \in A^S$ which occurs at x exactly once (say $\sigma^k(x) \in [p]$).

1. As x, y are indistinguishable, p also occurs exactly once on y, say $\sigma^m(y) \in [p]$.
2. Let $(S_n)_{n \in \mathbb{N}}$ with $S_n \nearrow \mathbb{Z}^d$ and $S \subset S_n$. Let $p_n = \sigma^k(x)|_{S_n}$

Basic properties of indistinguishable pairs:

Let (x, y) be a non-trivial indistinguishable asymptotic pair. If x is not recurrent, then x and y lie on the same orbit.

Proof: Suppose x is not recurrent. Then there exists $p \in A^S$ which occurs at x exactly once (say $\sigma^k(x) \in [p]$).

1. As x, y are indistinguishable, p also occurs exactly once on y, say $\sigma^m(y) \in [p]$.
2. Let $(S_n)_{n\in\mathbb{N}}$ with $S_n \nearrow \mathbb{Z}^d$ and $S \subset S_n$. Let $p_n = \sigma^k(x)|_{S_n}$
3. By definition $\sigma^k(x) \in [p_n]$. Also, this n is unique. By indistinguishability, we must have $\sigma^m(y) \in [p_n]$.

Basic properties of indistinguishable pairs:

Let (x, y) be a non-trivial indistinguishable asymptotic pair. If x is not recurrent, then x and y lie on the same orbit.

Proof: Suppose x is not recurrent. Then there exists $p \in A^S$ which occurs at x exactly once (say $\sigma^k(x) \in [p]$).

1. As x, y are indistinguishable, p also occurs exactly once on y, say $\sigma^m(y) \in [p]$.
2. Let $(S_n)_{n \in \mathbb{N}}$ with $S_n \nearrow \mathbb{Z}^d$ and $S \subset S_n$. Let $p_n = \sigma^k(x)|_{S_n}$
3. By definition $\sigma^k(x) \in [p_n]$. Also, this n is unique. By indistinguishability, we must have $\sigma^m(y) \in [p_n]$.
4. As $\bigcap_{n \in \mathbb{N}} [p_n] = \sigma^k(x)$, we conclude that $\sigma^k(x) = \sigma^m(y)$.

The case of $\mathbb{Z}$

On $\mathbb{Z}$ life is easier (as opposed to $\mathbb{Z}^d$ with $d \geq 2$):

Let (x, y) be a non-trivial indistinguishable asymptotic pair. If a pattern p occurs in x, then it occurs intersecting their difference set.

The case of $\mathbb{Z}$

On $\mathbb{Z}$ life is easier (as opposed to $\mathbb{Z}^d$ with $d \geq 2$):

> Let (x, y) be a non-trivial indistinguishable asymptotic pair. If a pattern p occurs in x, then it occurs intersecting their difference set.

$x =$

1	0	0	1	0	0	1	0	1	0	0	1

$y =$

1	0	0	1	0	1	0	0	1	0	0	1

The case of $\mathbb{Z}$

On $\mathbb{Z}$ life is easier (as opposed to $\mathbb{Z}^d$ with $d \geq 2$):

> Let (x, y) be a non-trivial indistinguishable asymptotic pair. If a pattern p occurs in x, then it occurs intersecting their difference set.

$x =$	1	0	0	1	0	0	1	0	1	0	0	1
$y =$	1	0	0	1	0	1	0	0	1	0	0	1

The case of $\mathbb{Z}$

On $\mathbb{Z}$ life is easier (as opposed to $\mathbb{Z}^d$ with $d \geq 2$):

> Let (x, y) be a non-trivial indistinguishable asymptotic pair. If a pattern p occurs in x, then it occurs intersecting their difference set.

$x =$ | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 1 |

$y =$ | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 |

The case of $\mathbb{Z}$

On $\mathbb{Z}$ life is easier (as opposed to $\mathbb{Z}^d$ with $d \geq 2$):

> Let (x, y) be a non-trivial indistinguishable asymptotic pair. If a pattern p occurs in x, then it occurs intersecting their difference set.

$x =$ | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 1 |

$y =$ | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 |

The case of $\mathbb{Z}$

On $\mathbb{Z}$ life is easier (as opposed to $\mathbb{Z}^d$ with $d \geq 2$):

Let (x, y) be a non-trivial indistinguishable asymptotic pair. If a pattern p occurs in x, then it occurs intersecting their difference set.

$$x = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ \hline\end{array}$$

$$y = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ \hline\end{array}$$

Corollary: If x, y are indistinguishable with difference set $F = [\![0, k-1]\!]$ then their word complexity satisfies

$$|\mathcal{L}_n(x)| \leq k + n - 1.$$

The case of $\mathbb{Z}$

Let (x, y) be a non-trivial indistinguishable asymptotic pair. If x is recurrent, then it is uniformly recurrent.

The case of $\mathbb{Z}$

Let (x, y) be a non-trivial indistinguishable asymptotic pair. If x is recurrent, then it is uniformly recurrent.

In particular, if (x, y) is an indistinguishable asymptotic pair, with x recurrent and difference set $F = \{0, 1\}$ then:

The case of $\mathbb{Z}$

Let (x, y) be a non-trivial indistinguishable asymptotic pair. If x is recurrent, then it is uniformly recurrent.

In particular, if (x, y) is an indistinguishable asymptotic pair, with x recurrent and difference set $F = \{0, 1\}$ then:

1. x, y are uniformly recurrent.

The case of $\mathbb{Z}$

Let (x, y) be a non-trivial indistinguishable asymptotic pair. If x is recurrent, then it is uniformly recurrent.

In particular, if (x, y) is an indistinguishable asymptotic pair, with x recurrent and difference set $F = \{0, 1\}$ then:

1. x, y are uniformly recurrent.
2. $|\mathcal{L}_n(x)| = |\mathcal{L}_n(y)| = n + 1$

The case of $\mathbb{Z}$

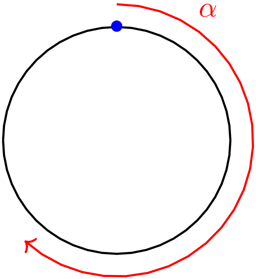
Let (x, y) be a non-trivial indistinguishable asymptotic pair. If x is recurrent, then it is uniformly recurrent.

In particular, if (x, y) is an indistinguishable asymptotic pair, with x recurrent and difference set $F = \{0, 1\}$ then:

1. x, y are uniformly recurrent.
2. $|\mathcal{L}_n(x)| = |\mathcal{L}_n(y)| = n + 1$

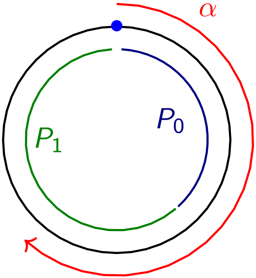
Thus x, y must be Sturmian configurations!

Let $\alpha \in [0,1] \setminus \mathbb{Q}$. Consider the rotation $R_\alpha \colon S^1 \to S^1$ given by $R_\alpha(x) = x + \alpha$.
Consider the partition $\mathcal{P} = \{P_0 = [0, 1-\alpha), [1-\alpha, 1)\}$.



$$\alpha = \frac{\sqrt{5}-1}{2}.$$

Let $\alpha \in [0,1] \setminus \mathbb{Q}$. Consider the rotation $R_\alpha \colon S^1 \to S^1$ given by $R_\alpha(x) = x + \alpha$.
Consider the partition $\mathcal{P} = \{P_0 = [0, 1-\alpha), [1-\alpha, 1)\}$.

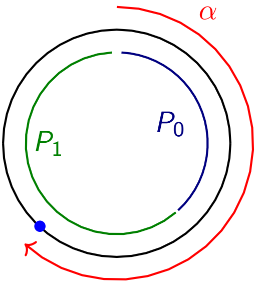


$\alpha = \frac{\sqrt{5}-1}{2}$.

$\varphi(x) = \ldots 0 \ldots$

Let $\alpha \in [0,1] \setminus \mathbb{Q}$. Consider the rotation $R_\alpha \colon S^1 \to S^1$ given by $R_\alpha(x) = x + \alpha$.

Consider the partition $\mathcal{P} = \{P_0 = [0, 1-\alpha), [1-\alpha, 1)\}$.

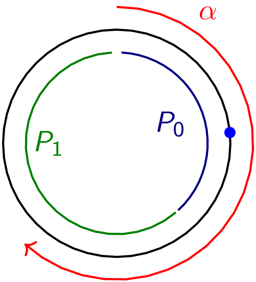


$\alpha = \frac{\sqrt{5}-1}{2}$.

$\varphi(x) = \ldots 01 \ldots$

Let $\alpha \in [0,1] \setminus \mathbb{Q}$. Consider the rotation $R_\alpha \colon S^1 \to S^1$ given by $R_\alpha(x) = x + \alpha$.

Consider the partition $\mathcal{P} = \{P_0 = [0, 1-\alpha), [1-\alpha, 1)\}$.

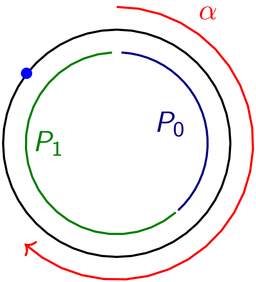


$$\alpha = \frac{\sqrt{5}-1}{2}.$$

$$\varphi(x) = \ldots 010 \ldots$$

Let $\alpha \in [0,1] \setminus \mathbb{Q}$. Consider the rotation $R_\alpha \colon S^1 \to S^1$ given by $R_\alpha(x) = x + \alpha$.

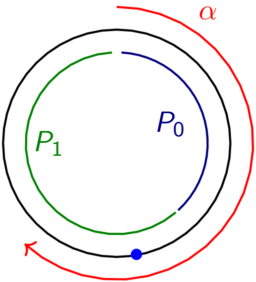
Consider the partition $\mathcal{P} = \{P_0 = [0, 1-\alpha), [1-\alpha, 1)\}$.



$\alpha = \frac{\sqrt{5}-1}{2}$.

$\varphi(x) = \ldots 0101 \ldots$

Let $\alpha \in [0,1] \setminus \mathbb{Q}$. Consider the rotation $R_\alpha \colon S^1 \to S^1$ given by $R_\alpha(x) = x + \alpha$.
Consider the partition $\mathcal{P} = \{P_0 = [0, 1-\alpha), [1-\alpha, 1)\}$.

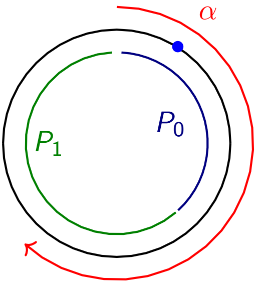


$$\alpha = \frac{\sqrt{5}-1}{2}.$$

$$\varphi(x) = \dots 01011 \dots$$

Let $\alpha \in [0,1] \setminus \mathbb{Q}$. Consider the rotation $R_\alpha \colon S^1 \to S^1$ given by $R_\alpha(x) = x + \alpha$.

Consider the partition $\mathcal{P} = \{P_0 = [0, 1-\alpha), [1-\alpha, 1)\}$.

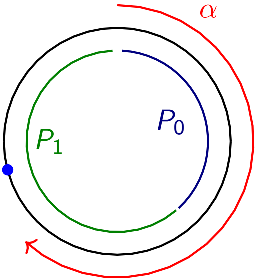


$\alpha = \frac{\sqrt{5}-1}{2}$.

$\varphi(x) = \ldots 010110 \ldots$

Let $\alpha \in [0,1] \setminus \mathbb{Q}$. Consider the rotation $R_\alpha \colon S^1 \to S^1$ given by $R_\alpha(x) = x + \alpha$.

Consider the partition $\mathcal{P} = \{P_0 = [0, 1-\alpha), [1-\alpha, 1)\}$.

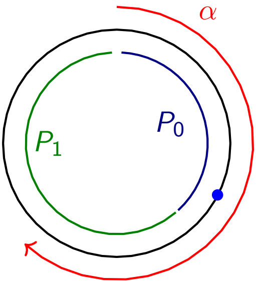


$$\alpha = \frac{\sqrt{5}-1}{2}.$$

$\varphi(x) = \ldots 0101101 \ldots$

Let $\alpha \in [0,1] \setminus \mathbb{Q}$. Consider the rotation $R_\alpha \colon S^1 \to S^1$ given by $R_\alpha(x) = x + \alpha$.

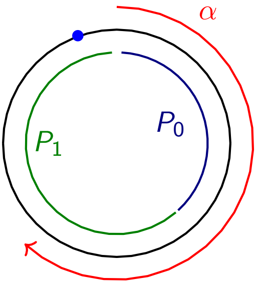
Consider the partition $\mathcal{P} = \{P_0 = [0, 1-\alpha), [1-\alpha, 1)\}$.



$\alpha = \frac{\sqrt{5}-1}{2}$.

$\varphi(x) = \dots 01011010 \dots$

Let $\alpha \in [0,1] \setminus \mathbb{Q}$. Consider the rotation $R_\alpha \colon S^1 \to S^1$ given by $R_\alpha(x) = x + \alpha$.
Consider the partition $\mathcal{P} = \{P_0 = [0, 1-\alpha), [1-\alpha, 1)\}$.

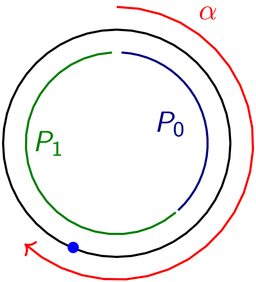


$\alpha = \frac{\sqrt{5}-1}{2}$.

$\varphi(x) = \ldots 010110101 \ldots$

Let $\alpha \in [0,1] \setminus \mathbb{Q}$. Consider the rotation $R_\alpha \colon S^1 \to S^1$ given by $R_\alpha(x) = x + \alpha$.

Consider the partition $\mathcal{P} = \{P_0 = [0, 1-\alpha), [1-\alpha, 1)\}$.



$$\alpha = \frac{\sqrt{5}-1}{2}.$$

$$\varphi(x) = \ldots 0101101011 \ldots$$

Formally, given $\alpha \in [0,1] \setminus \mathbb{Q}$ let $c_\alpha \in \{0,1\}^{\mathbb{Z}}$ be given by

$$c_\alpha(n) = \lfloor \alpha(n+1) \rfloor - \lfloor \alpha n \rfloor.$$

Formally, given $\alpha \in [0,1] \setminus \mathbb{Q}$ let $c_\alpha \in \{0,1\}^{\mathbb{Z}}$ be given by

$$c_\alpha(n) = \lfloor \alpha(n+1) \rfloor - \lfloor \alpha n \rfloor.$$

Choosing instead the partition $\mathcal{P}' = \{P_0 = (0, 1-\alpha], (1-\alpha, 1]\}$ gives

$$c'_\alpha(n) = \lceil \alpha(n+1) \rceil - \lceil \alpha n \rceil.$$

Formally, given $\alpha \in [0,1] \setminus \mathbb{Q}$ let $c_\alpha \in \{0,1\}^{\mathbb{Z}}$ be given by

$$c_\alpha(n) = \lfloor \alpha(n+1) \rfloor - \lfloor \alpha n \rfloor.$$

Choosing instead the partition $\mathcal{P}' = \{P_0 = (0, 1-\alpha], (1-\alpha, 1]\}$ gives

$$c'_\alpha(n) = \lceil \alpha(n+1) \rceil - \lceil \alpha n \rceil.$$

The pair (c_α, c'_α) is asymptotic with difference set $F = \{-1, 0\}$.

Formally, given $\alpha \in [0,1] \setminus \mathbb{Q}$ let $c_\alpha \in \{0,1\}^{\mathbb{Z}}$ be given by

$$c_\alpha(n) = \lfloor \alpha(n+1) \rfloor - \lfloor \alpha n \rfloor.$$

Choosing instead the partition $\mathcal{P}' = \{P_0 = (0, 1-\alpha], (1-\alpha, 1]\}$ gives

$$c'_\alpha(n) = \lceil \alpha(n+1) \rceil - \lceil \alpha n \rceil.$$

The pair (c_α, c'_α) is asymptotic with difference set $F = \{-1, 0\}$.

The pair (c_α, c'_α) is indistinguishable. In fact, every pattern in their language occurs exactly once intersecting each of their difference sets.

Theorem: B, Labbé and Starosta

Let $x, y \in \{0, 1\}^{\mathbb{Z}}$ and assume that x is recurrent. The following are equivalent:

- (x, y) is an indistinguishable asymptotic pair with difference set $F = \{-1, 0\}$ such that $x_{-1}x_0 = 10$ and $y_{-1}y_0 = 01$
- There exists $\alpha \in [0, 1] \setminus \mathbb{Q}$ such that $x = c_\alpha$ and $y = c'_\alpha$ are the lower and upper characteristic Sturmian sequences of slope α.

Theorem: B, Labbé and Starosta

Let $x, y \in \{0,1\}^{\mathbb{Z}}$ and assume that x is recurrent. The following are equivalent:

- (x, y) is an indistinguishable asymptotic pair with difference set $F = \{-1, 0\}$ such that $x_{-1}x_0 = 10$ and $y_{-1}y_0 = 01$
- There exists $\alpha \in [0,1] \setminus \mathbb{Q}$ such that $x = c_\alpha$ and $y = c'_\alpha$ are the lower and upper characteristic Sturmian sequences of slope α.

But there is more...

The non-recurrent case is an asymptotic limit of Sturmians.

Theorem: B, Labbé and Starosta

Let $x, y \in \{0,1\}^{\mathbb{Z}}$. The following are equivalent:

- (x, y) is an indistinguishable asymptotic pair with difference set $F = \{-1, 0\}$ such that $x_{-1}x_0 = 10$ and $y_{-1}y_0 = 01$
- there exists $(\alpha_n)_{n\in\mathbb{N}}$ with $\alpha_n \in [0,1] \setminus \mathbb{Q}$ such that

$$x = \lim_{n\to\infty} c_{\alpha_n} \quad \text{and} \quad y = \lim_{n\to\infty} c'_{\alpha_n}.$$

The non-recurrent case is an asymptotic limit of Sturmians.

Theorem: B, Labbé and Starosta

Let $x, y \in \{0,1\}^{\mathbb{Z}}$. The following are equivalent:

- (x, y) is an indistinguishable asymptotic pair with difference set $F = \{-1, 0\}$ such that $x_{-1}x_0 = 10$ and $y_{-1}y_0 = 01$
- there exists $(\alpha_n)_{n\in\mathbb{N}}$ with $\alpha_n \in [0,1] \setminus \mathbb{Q}$ such that

$$x = \lim_{n\to\infty} c_{\alpha_n} \quad \text{and} \quad y = \lim_{n\to\infty} c'_{\alpha_n}.$$

But there is more...

The general case can be obtained from Sturmians using shifts and substitutions.

Theorem: B, Labbé and Starosta

Let A be a finite alphabet and $x, y \in A^{\mathbb{Z}}$ a non-trivial asymptotic pair. Then x, y is indistinguishable if and only if either

- x is recurrent and there exists $\alpha \in [0,1] \setminus \mathbb{Q}$, a substitution $\varphi\colon \{0,1\} \to A^+$ and an integer $m \in \mathbb{Z}$ such that

$$\{x, y\} = \{\sigma^m \varphi(\sigma(c_\alpha)), \sigma^m \varphi(\sigma(c'_\alpha))\},$$

- x is not recurrent and there exists a substitution $\varphi\colon \{0,1\} \to A^+$ and an integer $m \in \mathbb{Z}$ such that

$$\{x, y\} = \{\sigma^m \varphi({}^\infty 0.10^\infty), \sigma^m \varphi({}^\infty 0.010^\infty)\}.$$

What about $d \geq 2$?

Things are much harder:

- Patterns may occur without intersecting the difference set.
- recurrent indistinguishable pairs may not be uniformly recurrent.
- Substitutions do not help reduce the problem to a small size difference set (no good notion of derived sequences).
- In general, there is no complexity bound.

Example:

x

2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
1	0	0	1	0	0	1	0	1	0	0	1
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2

y

2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
1	0	0	1	0	1	0	0	1	0	0	1
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2

The horizontal configuration is a 1-dimensional indistinguishable pair, everything else is the symbol 2.

Example:

x

2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
1	0	0	1	0	0	1	0	1	0	0	1
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2

y

2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
1	0	0	1	0	1	0	0	1	0	0	1
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2

The horizontal configuration is a 1-dimensional indistinguishable pair, everything else is the symbol 2.

- Recurrent but not uniformly recurrent.
- Some patterns do not occur in the difference set.

Theorem: B and Labbé.

Let $d \geq 1$ and $x, y \in \{0, \ldots, \mathtt{d}\}^{\mathbb{Z}^d}$ be an asymptotic pair with difference set $F = \{0, -e_1, \ldots, -e_d\}$. TFAE:

1. The asymptotic pair (x, y) is indistinguishable, satisfies the **flip condition** and x is uniformly recurrent.
2. There exists a totally irrational vector $\alpha \in [0, 1)^d$ such that $x = c_\alpha$ and $y = c'_\alpha$ are the **characteristic multidimensional Sturmian configurations** of slope α.

Theorem: B and Labbé.

Let $d \geq 1$ and $x, y \in \{0, \dots, \mathtt{d}\}^{\mathbb{Z}^d}$ be an asymptotic pair such that x is uniformly recurrent and which satisfies the **flip condition** with difference set $F = \{0, -e_1, \dots, -e_d\}$. TFAE:

1. The asymptotic pair (x, y) is indistinguishable.
2. For every nonempty finite connected subset $S \subset \mathbb{Z}^d$ and $p \in \mathcal{L}_S(x) \cup \mathcal{L}_S(y)$, p intersects the difference set F exactly once in both x and y.
3. For every nonempty finite connected subset $S \subset \mathbb{Z}^d$, we have
$$|\mathcal{L}_S(x)| = |\mathcal{L}_S(y)| = |F - S|.$$
4. There exists a totally irrational vector $\alpha \in [0, 1)^d$ such that $x = c_\alpha$ and $y = c'_\alpha$ are the **characteristic multidimensional Sturmian configurations** of slope α.

Multidimensional Sturmian Configurations

Let $(\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$ and consider the associated rotations $R_{\alpha_1}, \ldots, R_{\alpha_d}$.

Multidimensional Sturmian Configurations

Let $(\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$ and consider the associated rotations $R_{\alpha_1}, \ldots, R_{\alpha_d}$.

- If $(1, \alpha_1, \ldots, \alpha_d)$ is rationally independent, then the $\mathbb{Z}^d$-action on S^1 generated by the R_{α_i} is free.

Multidimensional Sturmian Configurations

Let $(\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$ and consider the associated rotations $R_{\alpha_1}, \ldots, R_{\alpha_d}$.

- If $(1, \alpha_1, \ldots, \alpha_d)$ is rationally independent, then the $\mathbb{Z}^d$-action on S^1 generated by the R_{α_i} is free.
- Consider the partition $\mathcal{W}$ of S^1 generated by refining the Sturmian partitions $\mathcal{P}_i = \{[0, 1-\alpha_i), [1-\alpha_i, 1)\}$ for every $1 \leq i \leq d$.

Multidimensional Sturmian Configurations

Let $(\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$ and consider the associated rotations $R_{\alpha_1}, \ldots, R_{\alpha_d}$.

- If $(1, \alpha_1, \ldots, \alpha_d)$ is rationally independent, then the $\mathbb{Z}^d$-action on S^1 generated by the R_{α_i} is free.
- Consider the partition $\mathcal{W}$ of S^1 generated by refining the Sturmian partitions $\mathcal{P}_i = \{[0, 1-\alpha_i), [1-\alpha_i, 1)\}$ for every $1 \leq i \leq d$.
- Respectively, let $\mathcal{W}'$ be the partition of S^1 generated by refining the Sturmian partitions $\mathcal{P}'_i = \{(0, 1-\alpha_i], (1-\alpha_i, 1]\}$ for every $1 \leq i \leq d$.

Multidimensional Sturmian Configurations

Let $(\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$ and consider the associated rotations $R_{\alpha_1}, \ldots, R_{\alpha_d}$.

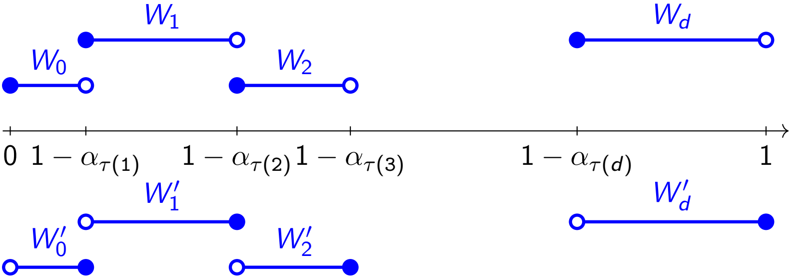
- If $(1, \alpha_1, \ldots, \alpha_d)$ is rationally independent, then the $\mathbb{Z}^d$-action on S^1 generated by the R_{α_i} is free.
- Consider the partition $\mathcal{W}$ of S^1 generated by refining the Sturmian partitions $\mathcal{P}_i = \{[0, 1-\alpha_i), [1-\alpha_i, 1)\}$ for every $1 \leq i \leq d$.
- Respectively, let $\mathcal{W}'$ be the partition of S^1 generated by refining the Sturmian partitions $\mathcal{P}'_i = \{(0, 1-\alpha_i], (1-\alpha_i, 1]\}$ for every $1 \leq i \leq d$.

The characteristic Sturmian configurations c_α, c'_α of slope α are the codings of 0 under the $\mathbb{Z}^d$-orbit generated by the rotations R_{α_i} and the partitions $\mathcal{W}$ and $\mathcal{W}'$ respectively.

Given $\alpha = (\alpha_1, \ldots, \alpha_d) \in [0,1]^d$, let $\tau \in S_d$ such that

$$1 \geq \alpha_{\tau(1)} \geq \alpha_{\tau(2)} \geq \cdots \geq \alpha_{\tau(d)} \geq 0.$$

Then the partitions $\mathcal{W}$ and $\mathcal{W}'$ are given by:



Explicitly, given $\alpha = (\alpha_1, \ldots, \alpha_d)$ we have

$$\begin{array}{rccl} c_\alpha : & \mathbb{Z}^d & \to & \{\mathtt{0}, \ldots, \mathtt{d}\} \\ & n & \mapsto & \sum_{i=1}^{d} \left(\lfloor \alpha_i + n \cdot \alpha \rfloor - \lfloor n \cdot \alpha \rfloor \right), \end{array}$$

and

$$\begin{array}{rccl} c'_\alpha : & \mathbb{Z}^d & \to & \{\mathtt{0}, \ldots, \mathtt{d}\} \\ & n & \mapsto & \sum_{i=1}^{d} \left(\lceil \alpha_i + n \cdot \alpha \rceil - \lceil n \cdot \alpha \rceil \right). \end{array}$$

Explicitly, given $\alpha = (\alpha_1, \ldots, \alpha_d)$ we have

$$\begin{array}{rcl} c_\alpha : \quad \mathbb{Z}^d & \to & \{\mathtt{0}, \ldots, \mathtt{d}\} \\ n & \mapsto & \sum_{i=1}^{d} \left(\lfloor \alpha_i + n \cdot \alpha \rfloor - \lfloor n \cdot \alpha \rfloor \right), \end{array}$$

and

$$\begin{array}{rcl} c'_\alpha : \quad \mathbb{Z}^d & \to & \{\mathtt{0}, \ldots, \mathtt{d}\} \\ n & \mapsto & \sum_{i=1}^{d} \left(\lceil \alpha_i + n \cdot \alpha \rceil - \lceil n \cdot \alpha \rceil \right). \end{array}$$

The configurations c_α, c'_α are asymptotic with difference set $F = \{0, -e_1, \ldots, -e_d\}$.

Recall the picture from the beginning:

x

1	0	2	2	1	0	2	1	0	2	1	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0	2	2	1	0
2	1	0	2	2	1	0	2	1	0	2	1	1	0	2
1	0	2	1	1	0	2	1	0	2	1	0	2	2	1
0	2	1	0	2	2	1	0	2	1	0	2	1	1	0
2	1	0	2	1	1	0	2	1	0	2	1	0	2	2
1	0	2	1	0	2	2	1	0	2	1	0	2	1	1
0	2	1	0	2	1	1	0	2	1	0	2	1	0	2
2	1	0	2	1	0	2	2	1	0	2	1	0	2	1
1	0	2	1	0	2	1	0	2	2	1	0	2	1	0
2	2	1	0	2	1	0	2	1	1	0	2	1	0	2
1	1	0	2	1	0	2	1	0	2	2	1	0	2	1
0	2	2	1	0	2	1	0	2	1	1	0	2	1	0
2	1	1	0	2	1	0	2	1	0	2	2	1	0	2
1	0	2	2	1	0	2	1	0	2	1	1	0	2	1

y

1	0	2	2	1	0	2	1	0	2	1	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0	2	2	1	0
2	1	0	2	2	1	0	2	1	0	2	1	1	0	2
1	0	2	1	1	0	2	1	0	2	1	0	2	2	1
0	2	1	0	2	2	1	0	2	1	0	2	1	1	0
2	1	0	2	1	1	0	2	1	0	2	1	0	2	2
1	0	2	1	0	2	2	1	0	2	1	0	2	1	1
0	2	1	0	2	1	0	2	2	1	0	2	1	0	2
2	1	0	2	1	0	2	1	1	0	2	1	0	2	1
1	0	2	1	0	2	1	0	2	2	1	0	2	1	0
2	2	1	0	2	1	0	2	1	1	0	2	1	0	2
1	1	0	2	1	0	2	1	0	2	2	1	0	2	1
0	2	2	1	0	2	1	0	2	1	1	0	2	1	0
2	1	1	0	2	1	0	2	1	0	2	2	1	0	2
1	0	2	2	1	0	2	1	0	2	1	1	0	2	1

We have $x = c_\alpha$ and $y = c'_\alpha$ respectively for

$$\alpha = \left(\frac{\sqrt{2}}{2}, \sqrt{19} - 4\right).$$

Flip Condition

Let $x, y \in \{0, \ldots, \mathtt{d}\}^{\mathbb{Z}^d}$ be an asymptotic pair. We say it satisfies the **flip condition** if:

1. the difference set of x and y is $F = \{0, -e_1, \ldots, -e_d\}$,
2. the restriction $x|_F$ is a bijection $F \to \{0, \ldots, \mathtt{d}\}$ such that $x_0 = 0$,
3. $y_n = x_n - 1 \mod (d+1)$ for every $n \in F$.

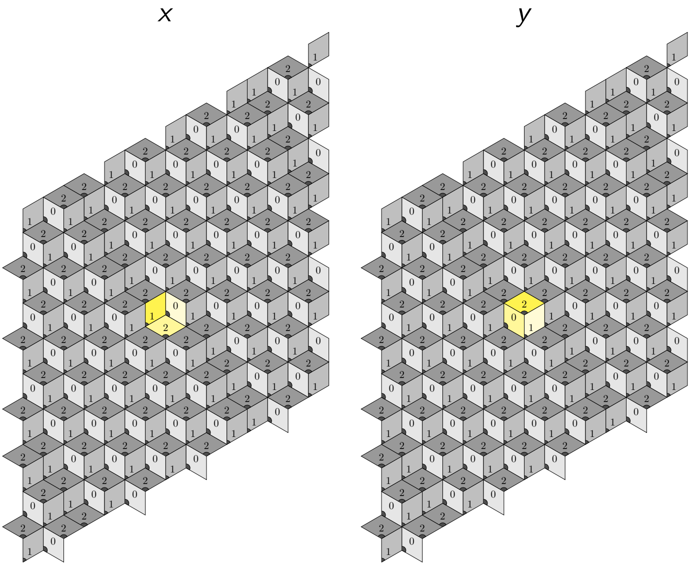
Flip Condition

Let $x, y \in \{0, \ldots, \mathtt{d}\}^{\mathbb{Z}^d}$ be an asymptotic pair. We say it satisfies the **flip condition** if:

1. the difference set of x and y is $F = \{0, -e_1, \ldots, -e_d\}$,
2. the restriction $x|_F$ is a bijection $F \to \{0, \ldots, \mathtt{d}\}$ such that $x_0 = 0$,
3. $y_n = x_n - 1 \bmod (d+1)$ for every $n \in F$.

The conditions above induce a permutation on $\{0, \ldots, \mathtt{d}\}$ defined by $y_n \mapsto x_n$ for every $n \in F$, which is the cyclic permutation $(0, 1, \ldots, d)$ of the alphabet.

The flip condition can be interpreted as flipping the unit hypercube on a co-dimension 1 discrete subspace.



Theorem: B and Labbé.

Let $d \geq 1$ and $x, y \in \{0, \ldots, \mathtt{d}\}^{\mathbb{Z}^d}$ be an asymptotic pair such that x is uniformly recurrent and which satisfies the **flip condition** with difference set $F = \{0, -e_1, \ldots, -e_d\}$. TFAE:

1. The asymptotic pair (x, y) is indistinguishable.
2. For every nonempty finite connected subset $S \subset \mathbb{Z}^d$ and $p \in \mathcal{L}_S(x) \cup \mathcal{L}_S(y)$, p intersects the difference set F exactly once in both x and y.
3. For every nonempty finite connected subset $S \subset \mathbb{Z}^d$, we have
$$|\mathcal{L}_S(x)| = |\mathcal{L}_S(y)| = |F - S|.$$
4. There exists a totally irrational vector $\alpha \in [0, 1)^d$ such that $x = c_\alpha$ and $y = c'_\alpha$ are the **characteristic multidimensional Sturmian configurations** of slope α.

For every nonempty finite connected subset $S \subset \mathbb{Z}^d$, we have

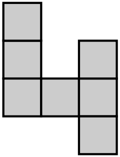
$$|\mathcal{L}_S(x)| = |\mathcal{L}_S(y)| = |F - S|.$$

For every nonempty finite connected subset $S \subset \mathbb{Z}^d$, we have

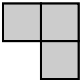
$$|\mathcal{L}_S(x)| = |\mathcal{L}_S(y)| = |F - S|.$$

Say $c_\alpha \in \{0, 1, 2\}^{\mathbb{Z}^d}$ and you need to know how many patterns with support $S \Subset \mathbb{Z}^2$ there are.

$S =$



$F =$

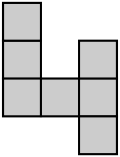


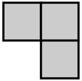
$F - S =$

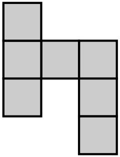
For every nonempty finite connected subset $S \subset \mathbb{Z}^d$, we have

$$|\mathcal{L}_S(x)| = |\mathcal{L}_S(y)| = |F - S|.$$

Say $c_\alpha \in \{0,1,2\}^{\mathbb{Z}^d}$ and you need to know how many patterns with support $S \Subset \mathbb{Z}^2$ there are.

$S =$ 

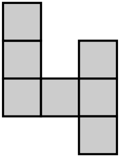
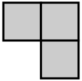
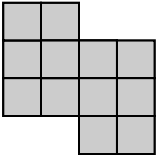
$F =$ 

$F - S =$ 

For every nonempty finite connected subset $S \subset \mathbb{Z}^d$, we have

$$|\mathcal{L}_S(x)| = |\mathcal{L}_S(y)| = |F - S|.$$

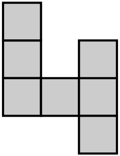
Say $c_\alpha \in \{0, 1, 2\}^{\mathbb{Z}^d}$ and you need to know how many patterns with support $S \Subset \mathbb{Z}^2$ there are.

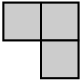
$S =$  $F =$  $F - S =$ 

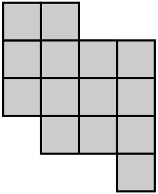
For every nonempty finite connected subset $S \subset \mathbb{Z}^d$, we have

$$|\mathcal{L}_S(x)| = |\mathcal{L}_S(y)| = |F - S|.$$

Say $c_\alpha \in \{0, 1, 2\}^{\mathbb{Z}^d}$ and you need to know how many patterns with support $S \Subset \mathbb{Z}^2$ there are.

$S =$ 

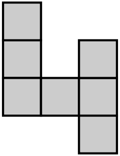
$F =$ 

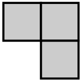
$F - S =$ 

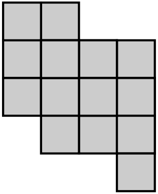
For every nonempty finite connected subset $S \subset \mathbb{Z}^d$, we have

$$|\mathcal{L}_S(x)| = |\mathcal{L}_S(y)| = |F - S|.$$

Say $c_\alpha \in \{0, 1, 2\}^{\mathbb{Z}^d}$ and you need to know how many patterns with support $S \Subset \mathbb{Z}^2$ there are.

$S =$ 

$F =$ 

$F - S =$ 

There are exactly 14 patterns with support S on a 2-dimensional Sturmian configuration.

Let $(m_1, \ldots, m_d) \in \mathbb{N}^d$ and consider the rectangle

$$R = \prod_{i=1}^{d} [\![0, m_i - 1]\!].$$

In this case we get a beautiful formula for the rectangular complexity of a multidimensional Sturmian configuration x:

$$|\mathcal{L}_R(x)| = |\mathcal{L}_{(m_1,\ldots,m_d)}(x)| = m_1 \cdots m_d \left(1 + \frac{1}{m_1} + \cdots + \frac{1}{m_d}\right).$$

Let $(m_1, \ldots, m_d) \in \mathbb{N}^d$ and consider the rectangle

$$R = \prod_{i=1}^{d} [\![0, m_i - 1]\!].$$

In this case we get a beautiful formula for the rectangular complexity of a multidimensional Sturmian configuration x:

$$|\mathcal{L}_R(x)| = |\mathcal{L}_{(m_1,\ldots,m_d)}(x)| = m_1 \cdots m_d \left(1 + \frac{1}{m_1} + \cdots + \frac{1}{m_d}\right).$$

We can interpret it as $|F - R|$, which is the volume of R, plus the volume of each of the $d-1$ dimensional faces.

Let $(m_1, \ldots, m_d) \in \mathbb{N}^d$ and consider the rectangle

$$R = \prod_{i=1}^{d} [\![0, m_i - 1]\!].$$

In this case we get a beautiful formula for the rectangular complexity of a multidimensional Sturmian configuration x:

$$|\mathcal{L}_R(x)| = |\mathcal{L}_{(m_1,\ldots,m_d)}(x)| = m_1 \cdots m_d \left(1 + \frac{1}{m_1} + \cdots + \frac{1}{m_d}\right).$$

We can interpret it as $|F - R|$, which is the volume of R, plus the volume of each of the $d-1$ dimensional faces.
$\triangleright$ For $d = 1$ we recover $L_n(x) = n + 1$.

Thanks!

Indistinguishable asymptotic pairs and multidimensional Sturmian configurations.
S. Barbieri, S. Labbé
https://arxiv.org/abs/2204.06413

A characterization of Sturmian sequences by indistinguishable asymptotic pairs
S. Barbieri, S. Labbé, Š. Starosta
https://doi.org/10.1016/j.ejc.2021.103318