Self-simulable groups

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Motivation

▷ Finitely presented group.

$$\Gamma = \langle s_1, \ldots, s_n \mid r_1, \ldots, r_k \rangle, \text{ with } r_i \in \{s_1, \ldots, s_n\}^*.$$

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▷ Finitely generated recursively presented group.

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Theorem (Higman 1961)

Every (finitely generated) recursively presented group occurs as a subgroup of a finitely presented group.

Topological Dynamical Systems

$$\Gamma \curvearrowright X$$

(Countable) group

Topological space
(compact metrizable)

Γ- action by homeomorphisms

Effective dynamical systems

Effective Dynamical Systems

$$\Gamma \curvearrowright X$$

(Finitely generated) group

Effective Γ- action

Effective space

Effective dynamical systems

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Effective $\leftrightarrow$ “Can be described through a Turing machine”

Setting

What would a dynamical analogue of Higman's theorem look like?

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Finitely presented group	**Recursively presented group**
$\updownarrow$	$\updownarrow$
• Subshift of finite type	• $\Gamma \curvearrowright X$ is an effectively closed action.

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What would a dynamical analogue of Higman's theorem look like?

Finitely presented group	**Recursively presented group**
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• Subshift of finite type	• $\Gamma \curvearrowright X$ is an effectively closed action.

$\triangleright$ In simpler words, we want a statement of the form: "every action which can be described by a Turing machine can be obtained in some nice way from a subshift of finite type."

Subshift of finite type

Let A be a finite set and consider $A^\Gamma = \{x \colon \Gamma \to A\}$ with the prodiscrete topology and the action $\Gamma \curvearrowright A^\Gamma$ given by

$$(gx)(h) = x(g^{-1}h) \text{ for every } g, h \in \Gamma.$$

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Subshift of finite type

A set $Y \subseteq A^\Gamma$ is a Γ-**subshift of finite type** (SFT) is there is a finite set $F \subseteq \Gamma$ and $\mathcal{F} \subseteq A^F$ such that $y \in Y$ if and only if

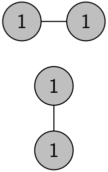
$$(gy)|_F \notin \mathcal{F} \text{ for every } g \in \Gamma.$$

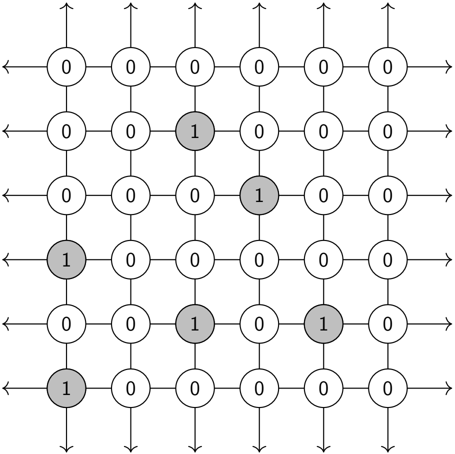
A subshift is of finite type if it is the set of configurations $x \in A^\Gamma$ which avoid a finite list of forbidden patterns (represented by $\mathcal{F}$).

Examples

Hard-square shift. $Z = \{x : \mathbb{Z}^2 \to \{0,1\}\}$ such that there are no vertical or horizontally adjacent 1s.

$\mathcal{F} =$

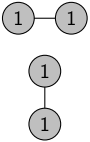


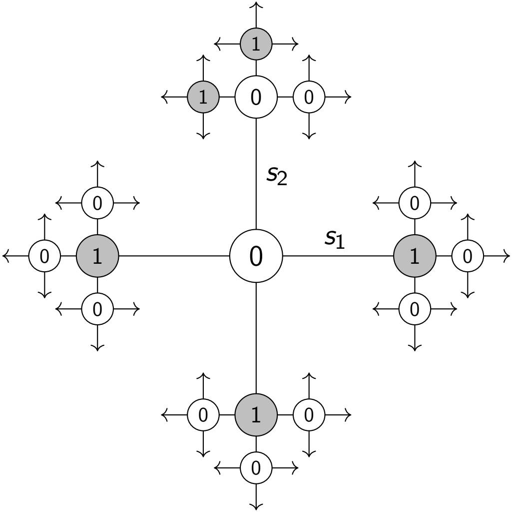


Examples

Hard-square in F_2.

$\mathcal{F} =$





Definitions

X can be described by a Turing machine

For a word $w = w_0 w_1 \ldots w_{n-1} \in \{0,1\}^n$ consider the cylinder set

$$[w] = \{x \in \{0,1\}^{\mathbb{N}} : x|_{\{0,\ldots,n-1\}} = w\}.$$

Effectively closed set

A set $X \subseteq \{0,1\}^{\mathbb{N}}$ is called **effectively closed** if it is closed and there is a Turing machine which enumerates a sequence of words $(w_n)_{n\in\mathbb{N}}$ such that

$$X = \{0,1\}^{\mathbb{N}} \setminus \bigcup_{n\in\mathbb{N}} [w_n].$$

Definitions

$\Gamma \curvearrowright X$ can be described by a Turing machine

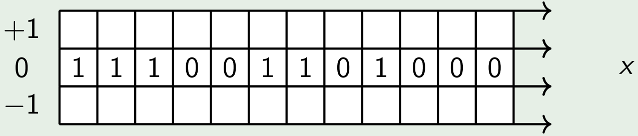
Let Γ be finitely generated by a symmetric set $S \ni 1_\Gamma$ and $X \subseteq \{0,1\}^{\mathbb{N}}$. Given $\Gamma \curvearrowright X$ consider the set

$$Y = \{y \in (\{0,1\}^S)^{\mathbb{N}} : \pi_s(y) = s \cdot \pi_{1_\Gamma}(y) \in X \text{ for every } s \in S\}.$$

Where $\pi_s(y) \in \{0,1\}^{\mathbb{N}}$ is such that $\pi_s(y)(n) = y(n)(s)$.

Example: $\mathbb{Z} \curvearrowright \{0,1\}^{\mathbb{N}}$ odometer

Let $S = \{-1, 0, +1\}$



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0	1	1	1	0	0	1	1	0	1	0	0	0	x
-1													

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$y \in Y$

Definitions

Effectively closed action

An action $\Gamma \curvearrowright X \subseteq \{0,1\}^{\mathbb{N}}$ is effectively closed if Y is an effectively closed set.

Intuitively: there is an algorithm telling me (1) when $x \notin X$ and (2) when $x \neq s \cdot y$.

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Note: In this talk we will always suppose that Γ has decidable word problem to avoid certain technicalities.

"Γ has decidable word problem if there's an algorithm that can *draw* arbitrarily large balls of its Cayley graph"

Example: natural actions of Thompson's groups

Consider $X = \{0,1\}^{\mathbb{N}}$ and let $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ be non-empty words in $\{0,1\}^*$ such that

$$X = [u_1] \sqcup [u_2] \sqcup \cdots \sqcup [u_n] = [v_1] \sqcup [v_2] \sqcup \cdots \sqcup [v_n].$$

Let φ be the homeomorphism of $\{0,1\}^{\mathbb{N}}$ which maps every cylinder $[u_i]$ to $[v_i]$ by replacing prefixes, that is

$$\varphi(u_i x) = v_i x \text{ for every } x \in \{0,1\}^{\mathbb{N}}.$$

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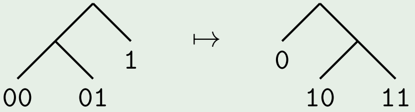
$u_1 = 00, u_2 = 01, u_3 = 1$ and $v_1 = 0, v_2 = 10, v_3 = 11$.

$$\varphi(0101010\dots) = 1001010\dots \quad \varphi(0000000\dots) = 0000000\dots$$
$$\varphi(1111111\dots) = 1111111\dots \quad \varphi(0011001\dots) = 011001\dots$$



Natural action of Thompson's groups

- F is the group of all such homeomorphisms where $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are given in lexicographical order.
- T is the group of all such homeomorphisms where $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are given in lexicographical order up to a cyclic permutation.
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$\triangleright$ these groups are finitely presented and have decidable word problem. Their natural action on $\{0,1\}^{\mathbb{N}}$ is effectively closed.

- T, V are nonamenable.
- It is a famous open problem whether F is amenable.

What results are known?

Hochman's theorem, 2009

Every effectively closed action $\mathbb{Z} \curvearrowright X$ is the topological factor of a subaction of a $\mathbb{Z}^3$-subshift of finite Z.

SFT $\quad \mathbb{Z}^3 \curvearrowright Z$

$$\begin{array}{ccc} \mathbb{Z}^3 \curvearrowright Z & & \\ \downarrow \text{subaction} & & \\ \mathbb{Z} \curvearrowright Z & \xrightarrow[\text{factor}]{} & \mathbb{Z} \curvearrowright X \end{array}$$

effective

topological factor = image under a continuous Γ-equivariant map

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$\mathbb{Z} \curvearrowright X \quad$ effective

topological factor $=$ image under a continuous Γ-equivariant map

Moreover, the factor is nice (mod a group rotation, 1-1 in a set of full measure with respect to any invariant measure.)

Similar results for actions of groups

B. Sablik, 2019

SFT $\mathbb{Z}^d \rtimes_\phi \Gamma \curvearrowright Z$

$d \geq 2$

$\phi \colon \Gamma \to \mathrm{GL}_d(\mathbb{Z})$

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B, 2019

SFT $\Gamma \times H \times V \curvearrowright Z$ $\qquad H, V$ infinite f.g.

subaction $\downarrow$

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A crazy question

Are there any groups Γ such that the diagram is as simple as possible?

Holy grail

$$\Gamma \curvearrowright Z \xrightarrow[\text{factor}]{} \!\!\!\!\!\!\rightarrow \Gamma \curvearrowright X$$

In words: are there finitely generated groups Γ such that every effectively closed action $\Gamma \curvearrowright X$ is the topological factor of a Γ-SFT Z?

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Theorem (B., Sablik, Salo 2021)

Yes.

Why is the question crazy?

Self-simulable group

A finitely generated group Γ is **self-simulable** if every effectively closed action $\Gamma \curvearrowright X$ is the topological factor of a Γ-SFT Z

A more proper name would be "groups with self-simulable zero-dimensional dynamics", but it is not that catchy.

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$\triangleright$ there are a lot of obstructions to self-simulability.

- Amenable groups cannot be self-simulable.
- Groups with infinitely many ends cannot be self-simulable.
- Some one-ended non-amenable groups are not self-simulable. Ex: $F_2 \times \mathbb{Z}$ (multi-ended $\times$ amenable).

Amenable groups

Let Γ be a group and let $K \subseteq \Gamma$ be finite and $\delta > 0$.

▷ we say $F \subseteq \Gamma$ is (K, δ)-invariant if

$$|KF \triangle F| \leq \delta |F|.$$

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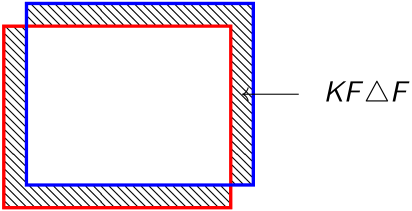

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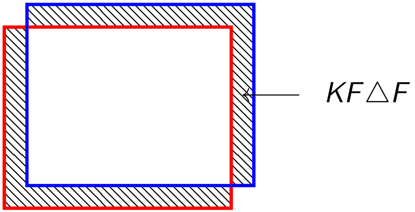

Amenable group

A group Γ is amenable if for every pair (K, δ) there exists a finite $F \subseteq \Gamma$ which is (K, δ)-invariant.

Amenable groups are not self-simulable

If Γ is amenable, we can associate to every action $\Gamma \curvearrowright X$ on a compact metrizable space by homeomorphisms a non-negative real number

$$h_{\text{top}}(\Gamma \curvearrowright X) \in [0, +\infty].$$

called the **topological entropy** of $\Gamma \curvearrowright X$.

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3. Conclusion: no action with entropy $+\infty$ can be the factor of a subshift.

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2. Topological entropy cannot increase under factors.
3. Conclusion: no action with entropy $+\infty$ can be the factor of a subshift.
4. If Γ is recursively presented, there are effectively closed actions $\Gamma \curvearrowright X$ with infinite entropy (the inverse limit of the full Γ-shifts on n symbols).

Our main result

Theorem (B., Sablik, Salo 2021)

The direct product $\Gamma = \Gamma_1 \times \Gamma_2$ of any pair of non-amenable finitely generated groups is self-simulable.

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- No need for self-similar or hierarchical structures as in the other results in the literature.
- Proof based on the existence of paradoxical decompositions.
- The technique is very flexible and allows for many other applications.

Amenable and non-amenable groups

- ▷ A non-amenable group is a group which is not amenable ☺

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Non-amenable group

A group Γ is non-amenable if and only if it admits a **paradoxical decomposition**.

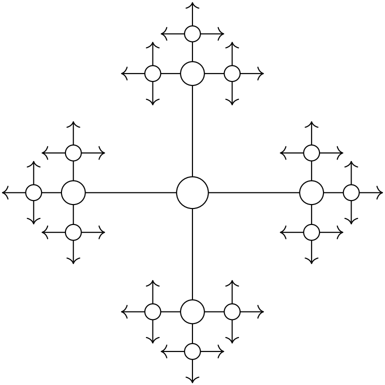
There is a partition $\Gamma = A \sqcup B$ and subpartitions

$$A = \bigsqcup_{i=1}^{n} A_i, \quad B = \bigsqcup_{j=1}^{k} B_j,$$

and elements $a_1, \ldots, a_n \in \Gamma$, $b_1, \ldots, b_k \in \Gamma$ such that

$$\Gamma = \bigsqcup_{i=1}^{n} a_i A_i = \bigsqcup_{j=1}^{k} b_j B_j.$$

Example: $F_2 = \langle a, b \rangle$.



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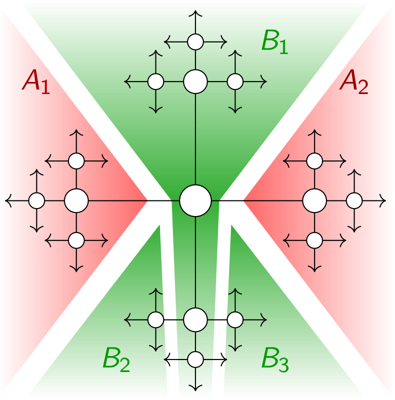

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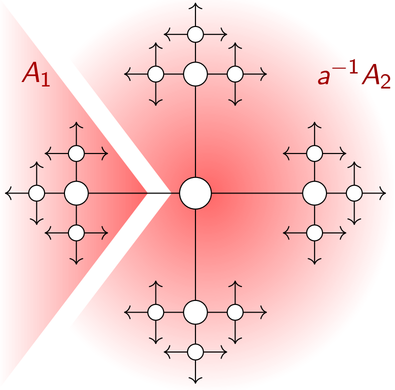

Amenable and non-amenable groups

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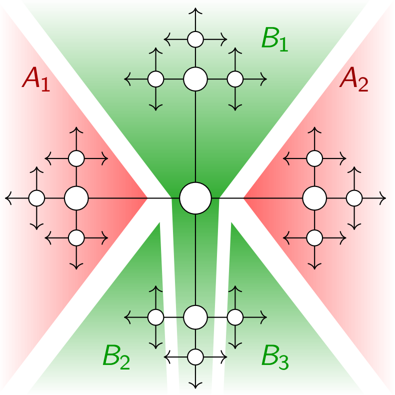

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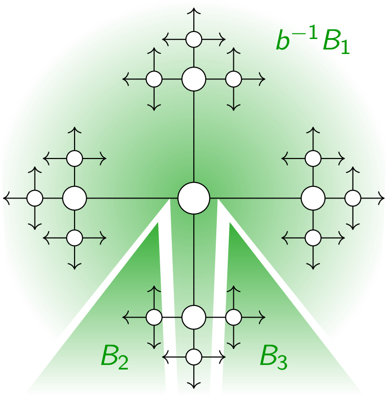

Amenable and non-amenable groups

- Paradoxical decompositions can be expressed analytically.

Non-amenable group

A group Γ is non-amenable if and only if there exists a finite set $K \subseteq \Gamma$ and a 2-to-1 map $\varphi \colon \Gamma \to \Gamma$ such that

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▷ The collection of all such maps can be coded using a Γ-subshift of finite type.

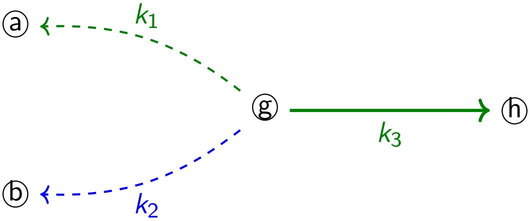
$$\textbf{Alphabet} = K^3 \times \{\mathtt{G}, \mathtt{B}\}.$$

- Three directions K^3: one pointing to $\varphi(g)$, the next two pointing to the two preimages
- A color (green or blue) (partitioning the elements of the group into two paradoxical sets).

The paradoxical subshift

In pictures, the alphabet represents the following structure.

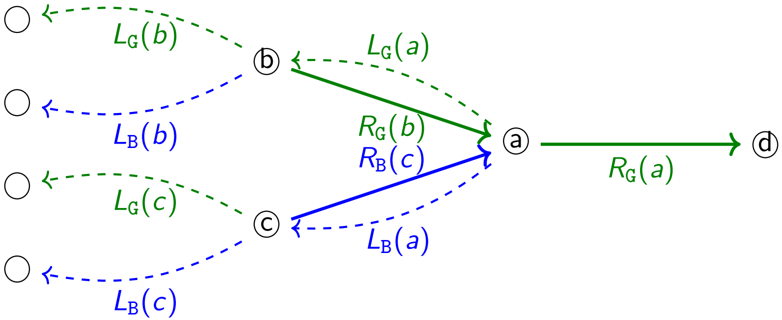
$$(k_1, k_2, k_3, \mathtt{G}) \in K^3 \times \{\mathtt{G}, \mathtt{B}\}$$



- $a \neq b$,
- $\varphi(a) = ak_1^{-1} = g$,
- $\varphi(b) = bk_2^{-1} = g$,
- $\varphi(g) = gk_3 = h$.

The paradoxical subshift

The local rules of the subshift impose that every node has two preimages of distinct color, and left arrows must match with right arrows.



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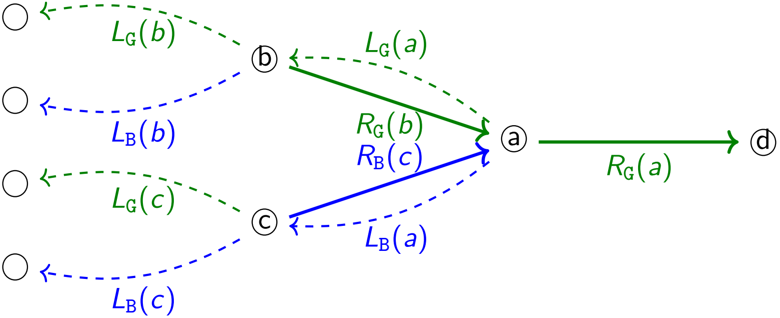
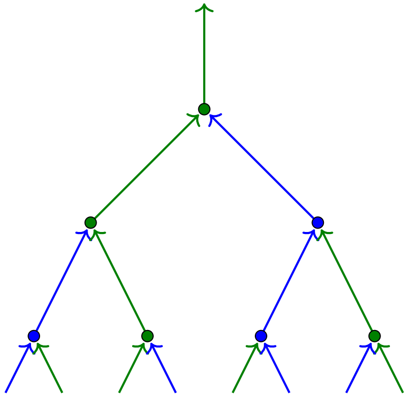


▷ This induces a binary tree structure.

▷ **Key observation**: In a bi-colored infinite binary tree, there is a canonical way to assign one-sided infinite paths to every node.



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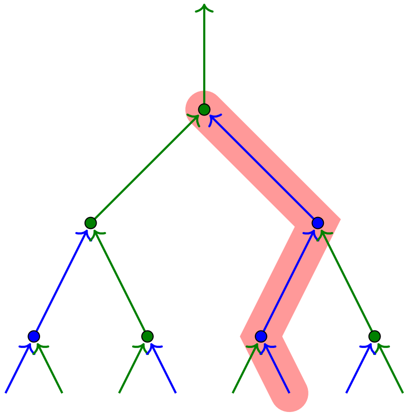

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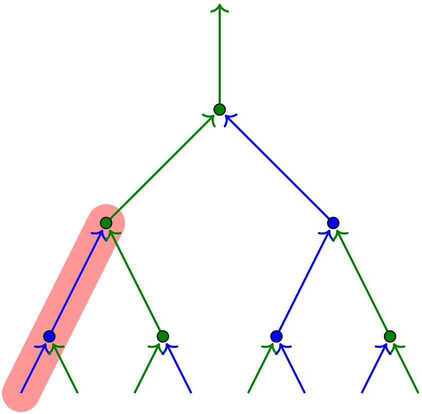

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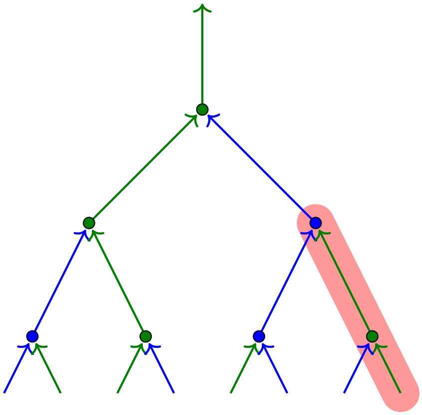

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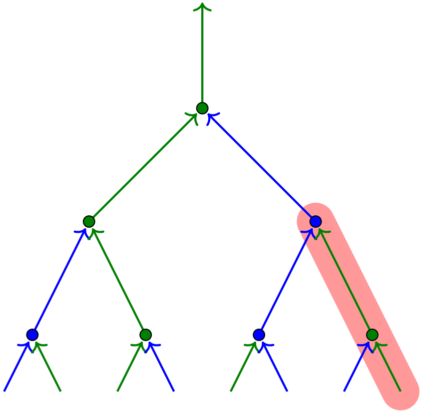

Follow the arrow tails of the opposite color!
The paths do not intersect.

The paradoxical shift

Lemma

In every non-amenable group Γ there is a Γ-subshift of finite type $\mathbf{P}$ called the **paradoxical shift** and a continuous function

$$\gamma \colon \mathbf{P} \times \mathbb{N} \times \Gamma \to \Gamma.$$

Such that for every $\rho \in \mathbf{P}$ the map

$$(n, g) \mapsto \gamma(\rho, n, g) \text{ for every } n \in \mathbb{N}, g \in \Gamma,$$

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In words: every configuration in the paradoxical shift encodes an assignment to every $g \in \Gamma$ of an infinite one-sided path with moves in a finite set $K \subseteq \Gamma$. Moreover, the paths do not intersect.

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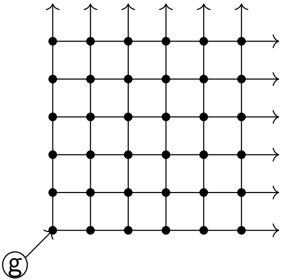

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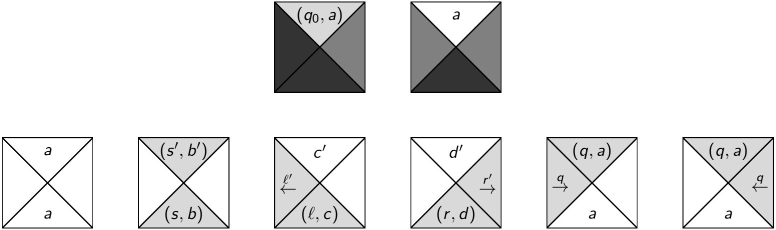

Where $\delta(s, b) = (s', b', 0)$, $\delta(\ell, c) = (\ell', c', -1)$ and $\delta(r, d) = (r', d', 1)$.

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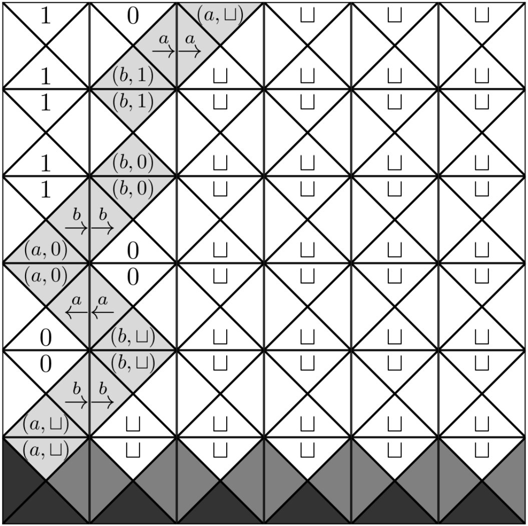

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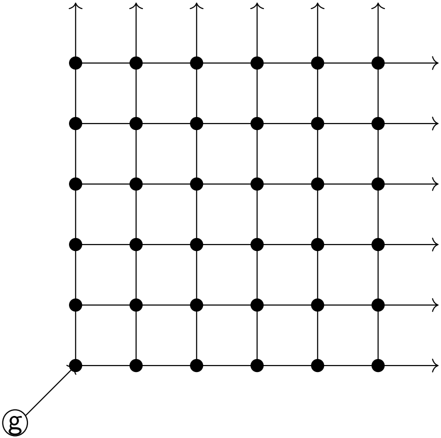
Result: The only remaining configurations are the ones in the set representation.

Start with $x = x_0 x_1 x_2 x_3 \cdots \in A^{\mathbb{N}}$



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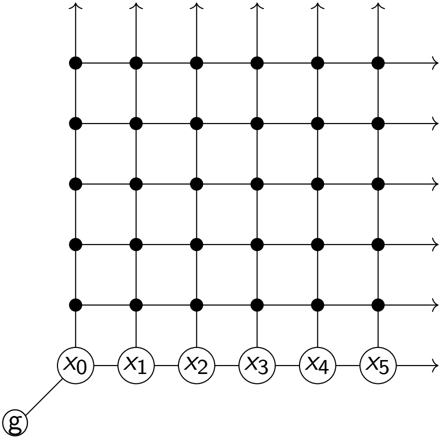

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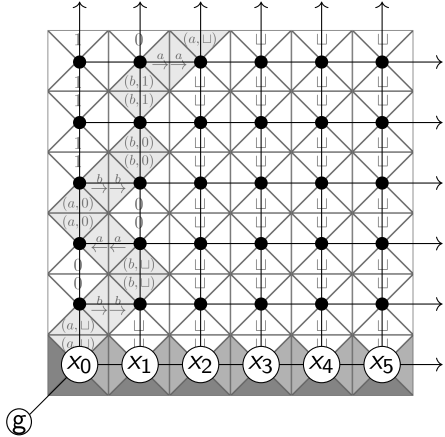

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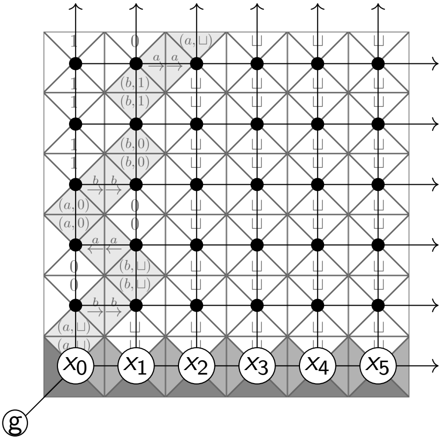

If the configuration survives (i.e. If the Turing machine does not stop), then x is in the set representation of $\Gamma \curvearrowright X$.

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Thus we obtain a natural factor map from this subshift of finite type to $\Gamma \curvearrowright X$.

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Theorem (B., Sablik, Salo 2021)

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Applications

Mixing the stability properties of this class, we obtain handy ways to show self-simulability:

Lemma

Let Γ be a finitely generated group which acts faithfully on $X = \{0,1\}^{\mathbb{N}}$ such that for any non-empty open set U the subgroup Γ_U which fixes every element of $X \setminus U$ is non-amenable. Then Γ is self-simulable.

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Theorem: Thompson's V is self-simulable

Proof: Consider the natural action $V \curvearrowright \{0,1\}^{\mathbb{N}}$ of Thompson's V. For any non-trivial word $w \in \{0,1\}^*$ the subgroup of V which fixes $X \setminus [w]$ is isomorphic to V (which is non-amenable).

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By a similar argument, if F is non-amenable then T is self-simulable.

Applications

The following groups are self-simulable:

- Finitely generated non-amenable branch groups.
- The finitely presented simple groups of Burger and Mozes.
- Thompson's group V and higher-dimensional Brin-Thompson's groups nV.
- The general linear groups $\mathrm{GL}_n(\mathbb{Z})$ and special linear groups $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 5$.
- The automorphism group $\mathrm{Aut}(F_n)$ and outter automorphism group $\mathrm{Out}(F_n)$ of the free group on at least $n \geq 5$ generators.
- Braid groups B_n on at least $n \geq 7$ strands.
- Right-angled Artin groups associated to the complement of a finite connected graph for which there are two edges at distance at least 3.

Applications

▷ Suppose $\Gamma \curvearrowright X$ admits a free effectively closed action (for every $x \in X$ then $gx = x$ implies that $g = 1_\Gamma$)

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Proof.

- Let $\phi \colon Z \to X$ be the factor map, and let $x \in Z$ and $g \in \Gamma$ such that $gx = x$.
- Then $g\phi(x) = \phi(gx) = \phi(x)$.
- As $\Gamma \curvearrowright X$ is free, we have $g = 1_\Gamma$. Thus $\Gamma \curvearrowright Z$ is free.

$\square$

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Corollary

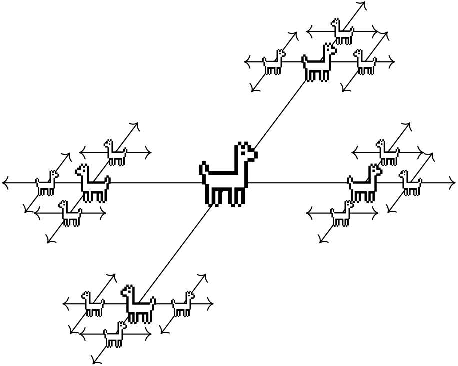
Every self-simulable group Γ with decidable word problem admits a Γ-SFT on which Γ acts freely.

Examples:

- $\Gamma = F_n \times F_n$.
- Thompson's V.
- Braid groups B_n, $n \geq 7$ strands.
- $GL_n(\mathbb{Z})$ and $SL_n(\mathbb{Z})$ for $n \geq 5$.

Note: If Γ is finitely generated, recursively presented and has undecidable word problem, there are no free effectively closed actions.

Thank you for your attention!



Groups with self-simulable zero-dimensional dynamics

S. Barbieri, M. Sablik and V. Salo

https://arxiv.org/abs/2104.05141