On the relation between topological entropy and asymptotic pairs

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June, 2020

Setting

- Topological dynamics.

$$\Gamma \curvearrowright X$$

- Γ: Countable (amenable) group
- $\curvearrowright$: Γ- action by homeomorphisms
- X: Compact metrizable space

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Two properties of $\Gamma \curvearrowright X$ that indicate the action is not too simple.

- Existence of asymptotic pairs $(x, y) \in X^2$

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Question

How are these two properties related?

Asymptotic pairs

Let d be a compatible metric on X.

Definition of asymptotic pairs

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$$d(gx, gy) \leq \varepsilon \text{ for every } g \in \Gamma \setminus F.$$

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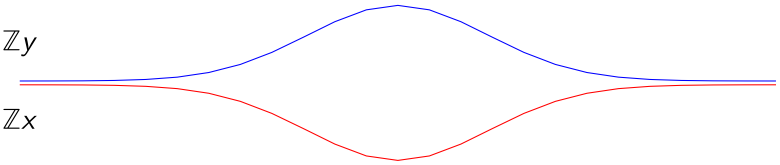
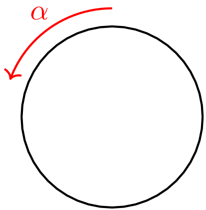


Examples

Let $X = \mathbb{R}/\mathbb{Z}$ and $\alpha \in \mathbb{R}$. Consider the rotation $\mathbb{Z} \curvearrowright \mathbb{R}/\mathbb{Z}$ given by

$$x \mapsto x + \alpha \bmod 1.$$



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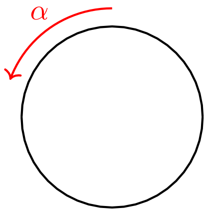

$$A(\mathbb{Z} \curvearrowright \mathbb{R}/\mathbb{Z}) = \{(x, x) : x \in \mathbb{R}/\mathbb{Z}\} = \triangle_{\mathbb{R}/\mathbb{Z}}.$$

For isometries, there are only trivial $(x = y)$ asymptotic pairs.

Examples

Let $X = \{0,1\}^\Gamma$ and $\Gamma \curvearrowright \{0,1\}^\Gamma$ be the shift action

$$gx(h) = x(g^{-1}h), \text{for every } g, h \in \Gamma, x \in \{0,1\}^\Gamma.$$

Then $(x, y) \in A(\Gamma \curvearrowright \{0,1\}^\Gamma)$ if and only if there is a finite $F \subseteq \Gamma$ such that

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$$\overline{\mathrm{A}(\Gamma \curvearrowright \{0,1\}^\Gamma)} = (\{0,1\}^\Gamma)^2.$$

For the full shift, asymptotic points are dense.

Examples

Let $X = \Gamma \cup \{\infty\}$ be the one-point compactification of Γ with the discrete topology. Consider the action $\Gamma \curvearrowright X$ given by

$$gx = \begin{cases} gh & \text{if } x = h \in \Gamma \\ \infty & \text{if } x = \infty \end{cases}.$$

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This action is conjugate to the sunny-side up subshift:

$$X_{\leq 1} = \{x \in \{0,1\}^{\Gamma} : \#\{g \in \Gamma : x(g) = 1\} \leq 1\}$$

by identifying $g \mapsto 1_{\{g\}}$ and $\infty \mapsto 0^{\Gamma}$.

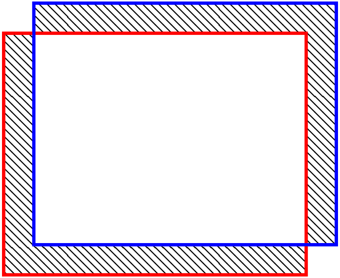
Topological entropy

Given $\delta > 0$ and a finite $K \subset \Gamma$, a set $F \subset \Gamma$ is called left (K, δ)-invariant if

$$|KF \triangle F| \leq \delta |F|$$

$F = [\![-n, n]\!]^2$

$K = \{(1,1)\}$



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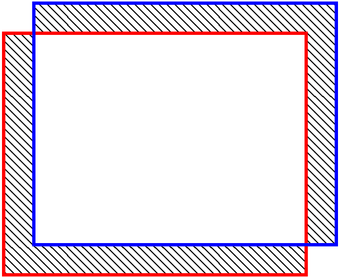

A group Γ is amenable if for every δ, K as above, there exists a finite set $F \subseteq \Gamma$ which is (K, δ)-invariant.

A sequence $\{F_n\}_{n \in \mathbb{N}}$ which is eventually (K, δ)-invariant for every K, δ is called Følner.

Let Γ be amenable, $\Gamma \curvearrowright X$ an action.
For an open cover $\mathcal{U}$ of X and $F \subseteq \Gamma$ let $\mathcal{U}^F$ be the refinement

$$\mathcal{U}^F = \bigvee_{g \in F} g^{-1}\mathcal{U},$$

and denote by $N(\mathcal{U})$ the minimum cardinality of a subcover.

The **topological entropy** of $\Gamma \curvearrowright X$ is given by

$$h_{\mathrm{top}}(\Gamma \curvearrowright X) = \sup_{\mathcal{U}} \lim \frac{1}{|F|} \log N(\mathcal{U}^F).$$

where the limit is taken as F becomes more and more left-invariant and the supremum is over all open covers of X.

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Main question

Under which conditions do we have that:

non-trivial asymptotic pairs imply positive entropy.

positive entropy implies non-trivial asymptotic pairs.

Non-trivial asymptotic pairs and zero entropy

FAIL

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- **(Boring example)** The sunny-side up subshift

$$X_{\leq 1} = \{x \in \{0,1\}^{\Gamma} : \#\{g \in \Gamma : x(g) = 1\} \leq 1\}$$

has zero topological entropy and $\mathrm{A}(\Gamma \curvearrowright X_{\leq 1}) = (X_{\leq 1})^2$.

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- **(Strictly ergodic example)** For $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, the Sturmian subshift associated to the rotation $x \mapsto x + \alpha \bmod 1$ has zero topological entropy and non-trivial asymptotic pairs.
For example, the codings x, y of 0 using the partitions

$$\{[0, 1-\alpha), [1-\alpha, 1)\} \text{ and } \{(0, 1-\alpha], (1-\alpha, 1]\}$$

respectively.

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- **(Meyerovitch 2017, [Theorem 1.3])** For every amenable residually finite group Γ, there is a Γ-subshift X with positive topological entropy and trivial asymptotic relation.

- ~~Topological~~ Algebraic dynamics.

$$\Gamma \curvearrowright X$$

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Compact metrizable ~~space~~ abelian group

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Example: toral automorphisms.

Remark: If (x, y) is asymptotic for an algebraic action, then $(y^{-1}x, e_X)$ is also asymptotic. The **homoclinic group of** $\Delta(\Gamma \curvearrowright X)$ of $\Gamma \curvearrowright X$ is the subgroup of $x \in X$ so that (x, e_X) is asymptotic.

Known results for algebraic dynamics

Theorem (Lind and Schmidt, 1999)

Let $\mathbb{Z}^d \curvearrowright X$ be an expansive algebraic action, then:

$$h_{\mathrm{top}}(\mathbb{Z}^d \curvearrowright X) > 0 \iff \Delta(\mathbb{Z}^d \curvearrowright X) \neq \{e_X\}.$$

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Theorem (Chung and Li, 2014)

Let $\Gamma \curvearrowright X$ be an expansive algebraic action and Γ be polycyclic-by-finite, then:

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A result for topological dynamics

Theorem (Meyerovitch, 2018)

Let $\Gamma \curvearrowright X$ be an expansive action of a countable amenable group which satisfies ***the pseudo-orbit tracing property****, then:*

$$h_{\text{top}}(\Gamma \curvearrowright X) > 0 \implies \textit{There are non-trivial asymptotic pairs}$$

The converse also holds if some non-trivial asymptotic pair is in the support of a Γ-invariant Borel probability measure.

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Remark: The pseudo-orbit tracing property (POTP) is also known as **shadowing**.

The POTP

A sequence $\{x_g\}_{g\in\Gamma}$ is an (S,δ)-pseudo orbit if

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An action $\Gamma \curvearrowright X$ has the **pseudo-orbit tracing property** (POTP) if for every $\varepsilon > 0$ there is a finite $S \subseteq \Gamma$ and $\delta > 0$ so that every (S, δ)-pseudo orbit $\{x_g\}_{g \in \Gamma}$ is ε-traced by some $y \in X$.

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Examples: hyperbolic toral automorphisms, subshifts of finite type.

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Meyerovitch $\not\Rightarrow$ Chung and Li ✗

[Bhattacharya, 2019] There exists an expansive algebraic action of a polycyclic group which does not have the POTP.

☹ we need to find a weaker property...

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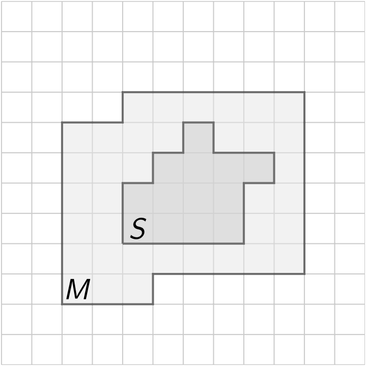

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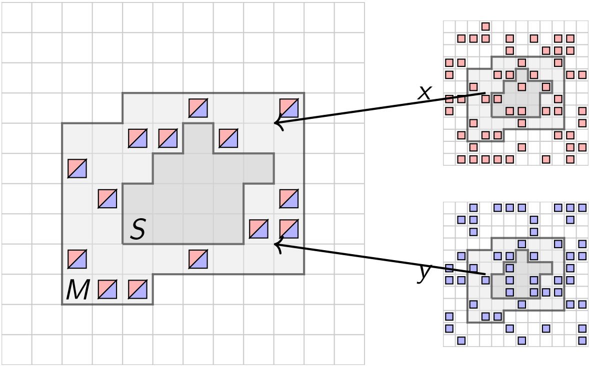

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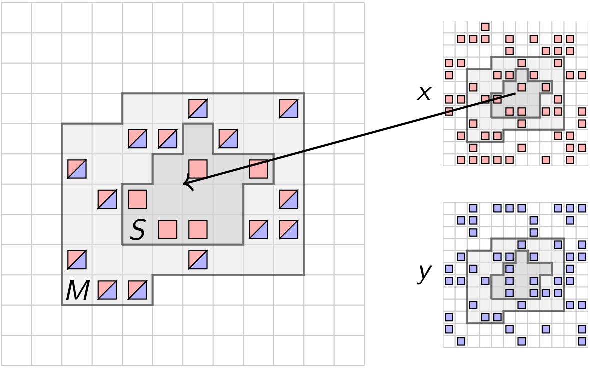

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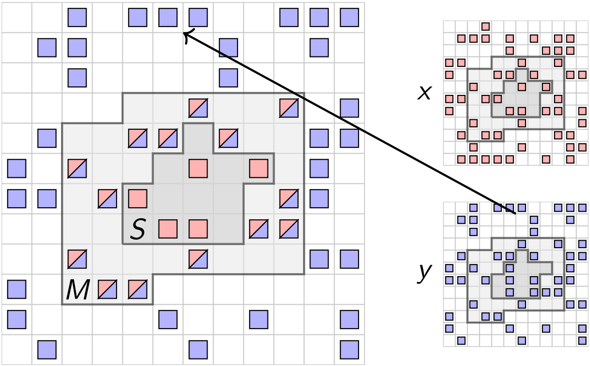

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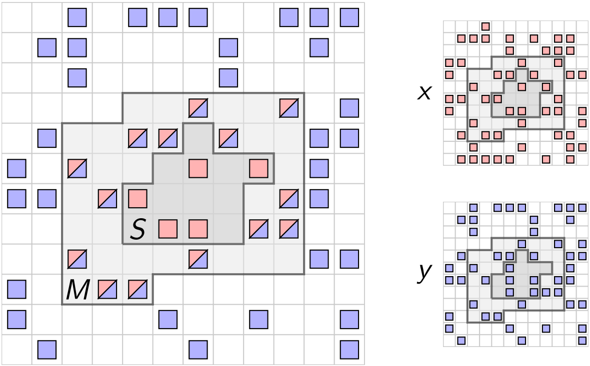

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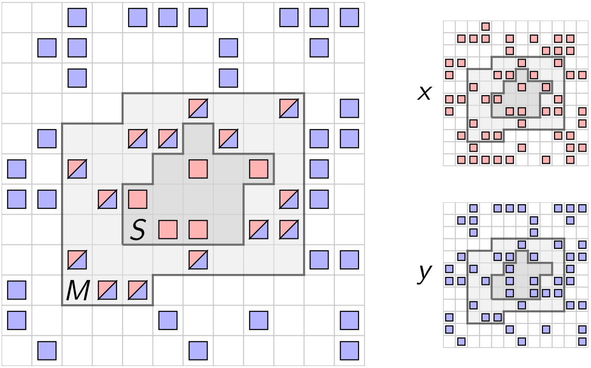

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$$\text{SFT} \longrightarrow \text{Strong TMP} \longrightarrow \text{TMP}$$

SFT	Strong TMP	TMP
symbolic actions with POTP	Bounded memory sets	Finite memory sets

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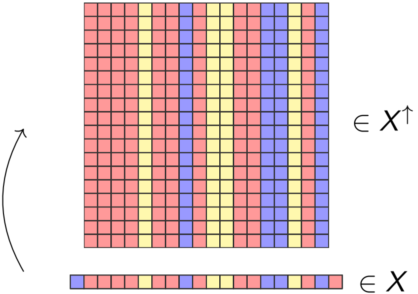
Example (trivial)

Let $X \subseteq A^\Gamma$ be a $\mathbb{Z}$-subshift. Let $X^\uparrow$ be its trivial extension to $\mathbb{Z}^2$:



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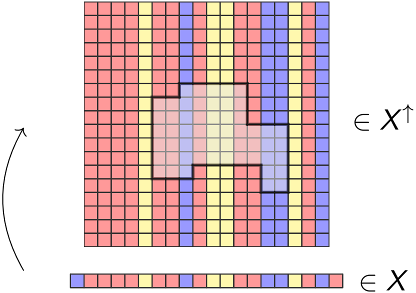

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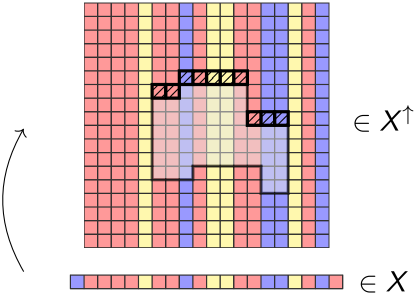

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- The class of $\mathbb{Z}^2$-subshifts with strong TMP is **uncountable**.
- All non-negative real numbers are top. entropies of subshifts with strong TMP. (Take $X =$ sturmian with slope α and split the symbol with measure α in $X^\uparrow$.)

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- Every expansive and finitely presented algebraic action of an amenable group which satisfies the **strong Atiyah conjecture** has the strong TMP.
  - Torsion-free elementary amenable groups.
  - Left-orderable amenable groups.

Back to Tom's theorem

Theorem (Meyerovitch, 2018)

Let Γ *be amenable and* $\Gamma \curvearrowright X$ *expansive with POTP, then:*

$$h_{\text{top}}(\Gamma \curvearrowright X) > 0 \implies \textit{There are non-trivial asymptotic pairs}$$

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Let Γ be amenable and $\Gamma \curvearrowright X$ be an expansive action then:

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Local entropy theory

- Entropy is a global notion $\leftrightsquigarrow$ asymptotic pairs is a local notion.

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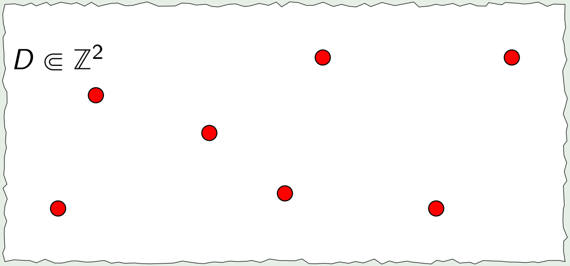
$(x, y) \in X^2$ is an **independence entropy pair** (IE-pair), if for every pair of open neighborhoods U_x, U_y of x and y there exists a set $D \subseteq \Gamma$ of *positive density* such that for every finite $I \subseteq D$ and function $\varphi \colon I \to \{x, y\}$ such that:

$$\bigcap_{g \in I} g^{-1} U_{\varphi(g)} \neq \varnothing.$$

D has positive density if for some Følner squence $\{F_n\}_{n\in\mathbb{N}}$ we have

$$\lim_{n\to\infty} \frac{|F_n \cap D|}{|F|} > 0.$$

Example: Let $x, y \in \{\square, \blacksquare\}^{\mathbb{Z}^2}$ such that  and  occurs in x and y respectively. If (x, y) is an IE-pair, then:



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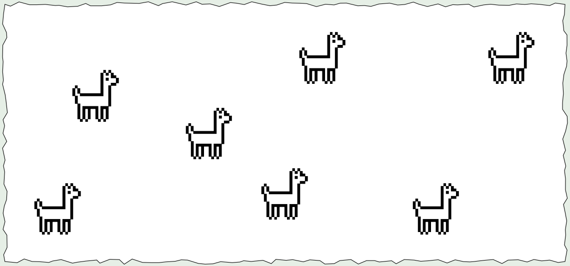

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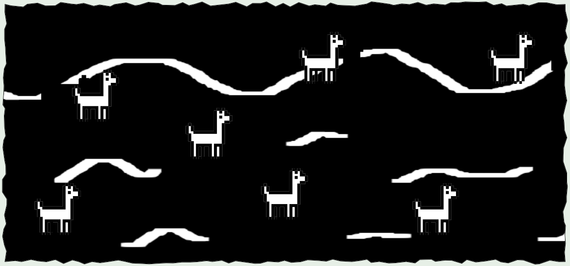

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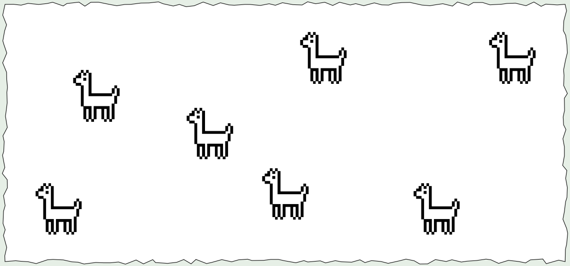

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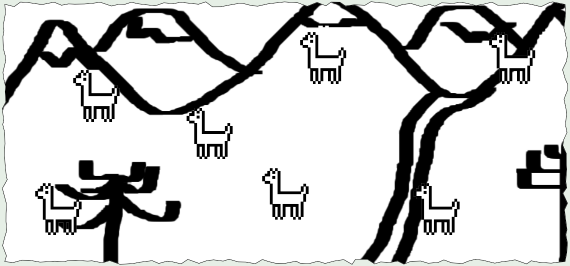

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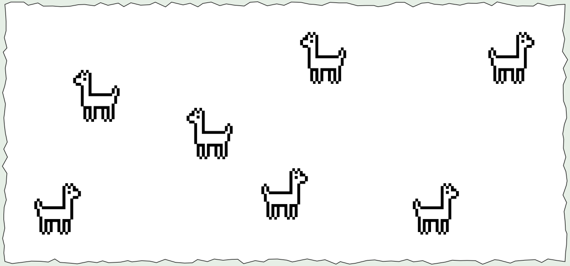

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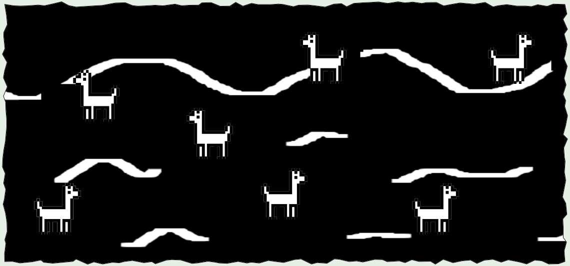

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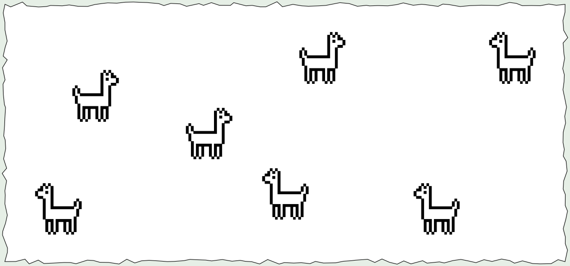

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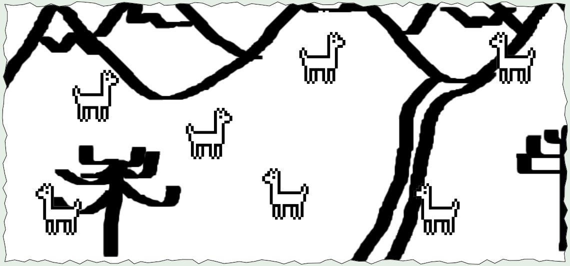

Theorem [Blanchard, 1993], [Kerr Li, 2007]

$h_{\mathrm{top}}(\Gamma \curvearrowright X) > 0$ if and only if there is an IE-pair (x, y) with $x \neq y$.

Proof sketch I

Let us prove it for a subshift $X \subseteq A^\Gamma$.
Suppose the entropy is positive and X has strong TMP.

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- As (x, y) is an IE-pair, there exists a positive density set $D \subseteq \Gamma$ so that for every finite $I \subseteq D$ and $\varphi \colon I \to \{x, y\}$ there is $z \in X$ such that for every $g \in I$

$$(g \cdot z)|_S = p_{\varphi(g)}.$$

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- There is n sufficiently large such that

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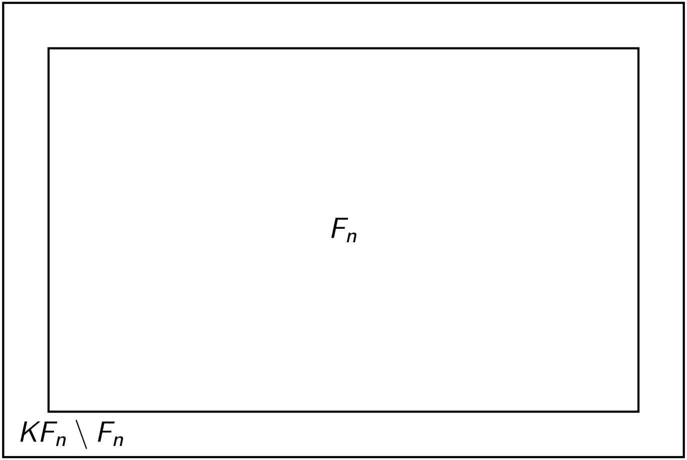
Proof sketch III



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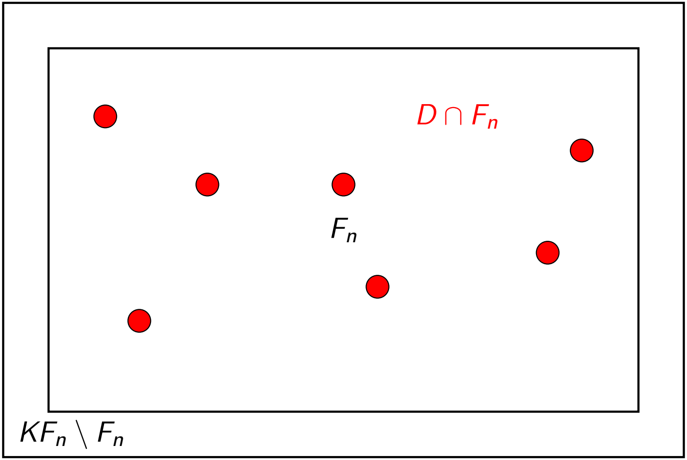

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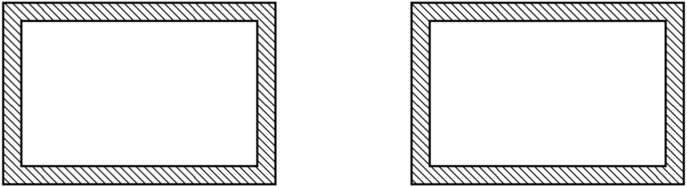

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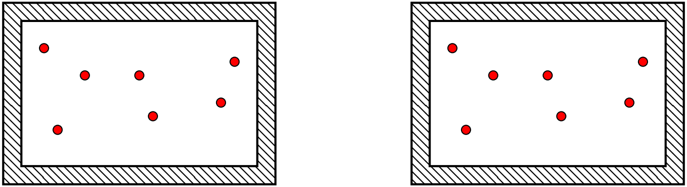

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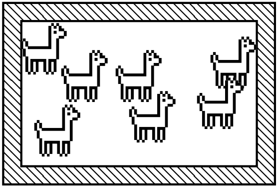
φ_1



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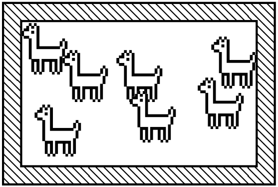

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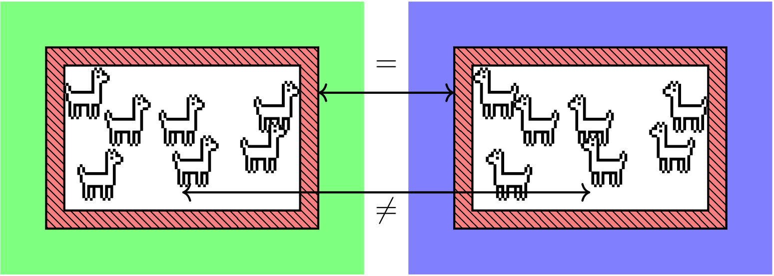

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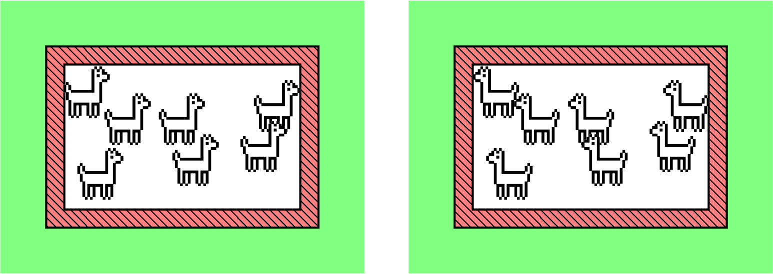

And we have our non-trivial asymptotic pair !

Proof sketch: converse I

Suppose X has the TMP and there is a non-trivial asymptotic pair in the support of some Γ-invariant measure μ.

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- Take $f = 1_{[p_x]}$.

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- We obtain that there exists a generic point z such that:

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$$h_{\mathrm{top}}(\Gamma \curvearrowright X) > 0.$$

Some applications

Let $X \subseteq A^\Gamma$ be a subshift. A measure μ on X is Markovian if there is a finite $F \subseteq \Gamma$ such that for every finite $S \subseteq \Gamma$ and $p \in A^S$

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Corollary [B, García Ramos, Li]

Let μ is a Markovian measure on X

- If $h_\mu(\Gamma \curvearrowright X) > 0$, then $\Gamma \curvearrowright \mathrm{supp}(\mu)$ has non-trivial asymptotic pairs.
- If $\Gamma \curvearrowright (X, \mu)$ is K, then the asymptotic pairs of $\Gamma \curvearrowright \mathrm{supp}(\mu)$ are dense in $\mathrm{supp}(\mu) \times \mathrm{supp}(\mu)$.

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It is known from Quas and Trow that every minimal $\mathbb{Z}^d$-SFT has zero topological entropy.

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In particular, minimal SFTs on amenable groups have zero entropy.

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- $\Delta(\Gamma \curvearrowright X) \subseteq IE(\Gamma \curvearrowright X)$.
- If $\Delta(\Gamma \curvearrowright X) \neq \varnothing$, then $h_{\mathrm{top}}(\Gamma \curvearrowright X) > 0$.

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Corollary

Let $\Gamma \curvearrowright X$ be an expansive algebraic action such that either

- $\mathbb{Z}\Gamma$ is left Noetherian or,
- Γ satisfies the strong Atiyah conjecture and the dual $\mathbb{Z}\Gamma$-module of $\Gamma \curvearrowright X$ is finitely presented.

Then $\overline{\Delta(\Gamma \curvearrowright X)} = IE(\Gamma \curvearrowright X)$.

Thank you for your attention!

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