Topological Markov properties

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Setup

Let $\Gamma \curvearrowright A^{\Gamma}$ be the shift action where:

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- A is a finite set (ex : $A = \{0, 1\}$)
- The action is given by

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A subshift which can be defined by a **finite** set of forbidden patterns is a **subshift of finite type (SFT)**.

Weakening the SFT condition

SFTs are defined by a **finite set of data**. A way to weaken the condition is to consider larger classes given by finite data.

SFT ⟶ Sofic ⟶ Effective

SFT	Sofic	Effective
finite set of patterns	SFT + factor map	effective set of patterns

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In this talk, we shall consider another way to weaken the SFT condition which is more dynamical in nature. The **topological Markov properties** (TMP).

SFT ⟶ Strong TMP ⟶ TMP

symbolic actions with POTP

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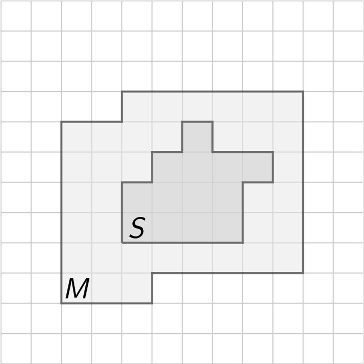

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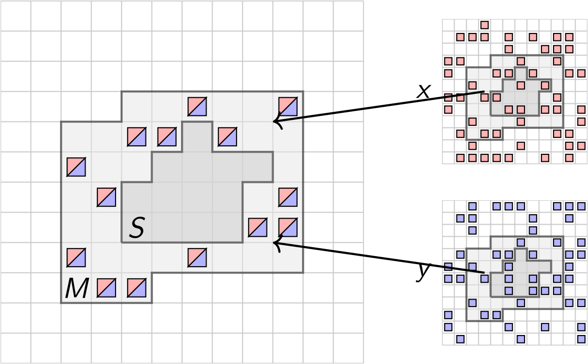

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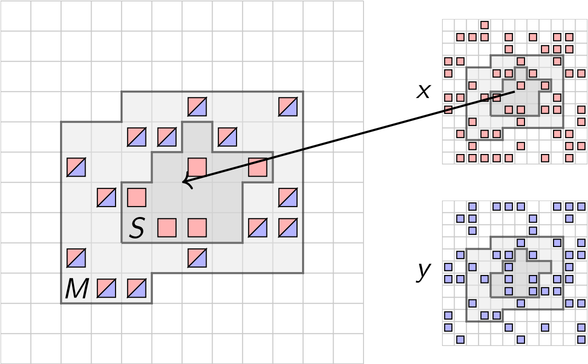

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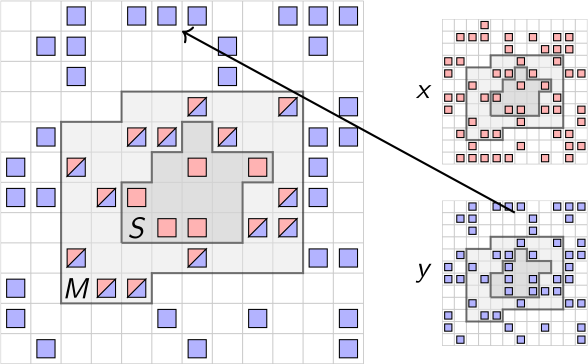

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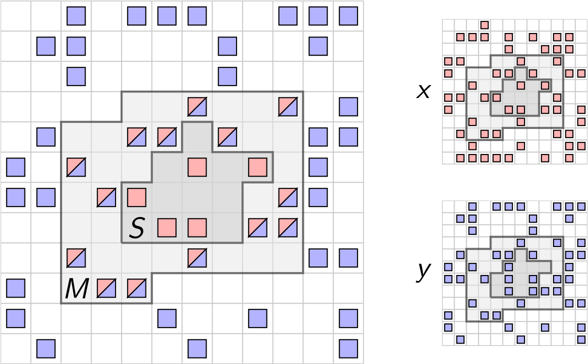

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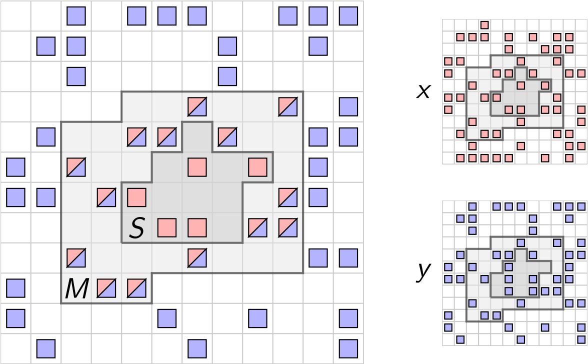

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Strong TMP is a bounded version of TMP.

$$\begin{tikzcd} \text{SFT} \arrow[r] & \text{Strong TMP} \arrow[r] & \text{TMP} \end{tikzcd}$$

symbolic actions with POTP	Bounded memory sets	Finite memory sets

Topological Markov property

Why a weakening of POTP?

- A sequence $\{x_g\}_{g\in\Gamma}$ is an (S,δ)-pseudo orbit if for every $s\in S$, $g\in\Gamma$
$$d(sx_g, x_{sg}) \leq \delta.$$
- An action $\Gamma \curvearrowright X$ has the **pseudo-orbit tracing property** (POTP) if for every $\varepsilon > 0$ there is a finite $S \subseteq \Gamma$ and $\delta > 0$ so that every (S,δ)-pseudo orbit $\{x_g\}_{g\in\Gamma}$ is ε-traced by some $y \in X$.
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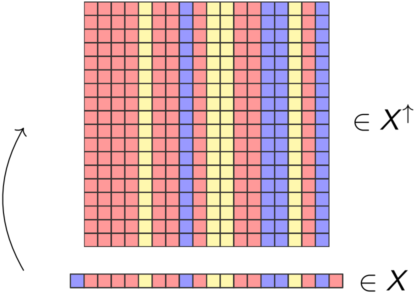
A subshift has POTP if and only if it is an SFT.

TMP $=$ restricted POTP.

Remark: TMP can be extended to actions $\Gamma \curvearrowright X$ by homeomorphisms on a compact metrizable space X. [not necessarily expansive nor zero-dimensional.]

Example (trivial)

Let $X \subseteq A^\Gamma$ be a $\mathbb{Z}$-subshift. Let $X^\uparrow$ be its trivial extension to $\mathbb{Z}^2$:



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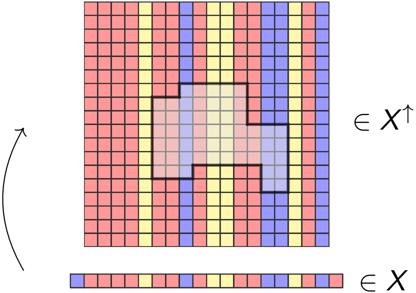

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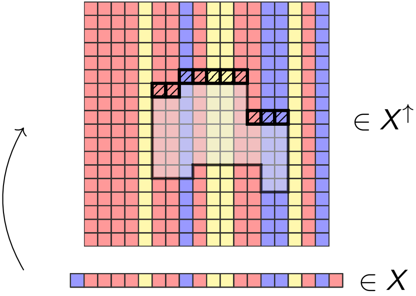

$X^{\uparrow}$ has the strong TMP with $F = \{(0,1)\}$.

- The class of $\mathbb{Z}^2$-subshifts with strong TMP is **uncountable**.
- All non-negative real numbers are top. entropies of subshifts with strong TMP. (Take X = sturmian with slope α and split the symbol with measure α in $X^{\uparrow}$.)

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- Every action $\Gamma \curvearrowright X$ on a compact metrizable group X by continuous automorphisms has the TMP. (ex: group shifts)
- Every expansive algebraic action of a polycyclic-by-finite group has the strong TMP (they do not have POTP!).
- Every expansive and finitely presented algebraic action of an amenable group which satisfies the **strong Atiyah conjecture** has the strong TMP.
  - Torsion-free elementary amenable groups.
  - Left-orderable amenable groups.

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The classes are much larger, and many classical theorems still hold in this setting. Furthermore, the proofs are “more natural”.

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The Lanford-Ruelle theorem states that for all $\mathbb{Z}^d$-SFTs then:

$$\{\texttt{Equilibrium measures}\} \subseteq \{\texttt{Gibbs measures}\}.$$

Theorem (B, Gómez, Marcus, Taati)

Let Γ be amenable and $X \subseteq A^\Gamma$ a subshift with the TMP, then:

$$\{\texttt{Equilibrium measures}\} \subseteq \{\texttt{Gibbs measures}\}.$$

Why should you care? (2/3)

It is known from Quas and Trow that every minimal $\mathbb{Z}^d$-SFT has zero topological entropy.

- The proof uses strongly that $\mathbb{Z}^d$ is left-orderable.
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Theorem (B, García Ramos, Li)

Let Γ be amenable, and $X \subseteq A^{\Gamma}$ a minimal subshift with strong TMP. Then X has zero topological entropy.

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Zero-dimensional is not needed: Let Γ be amenable and $\Gamma \curvearrowright X$ a minimal expansive action with the strong TMP. Then $\Gamma \curvearrowright X$ has zero topological entropy

Why should you care? (3/3)

Theorem [Meyerovitch, 2017] Let Γ be amenable and $\Gamma \curvearrowright X$ be an expansive action with POTP. Then:

$\Gamma \curvearrowright X$ has positive entropy $\iff$ $\Gamma \curvearrowright X$ admits off-diagonal asymptotic pairs in the support of an invariant measure.

Theorem (B, García Ramos, Li)

Let Γ be amenable and $\Gamma \curvearrowright X$ be an expansive action then:

- *If $\Gamma \curvearrowright X$ has* ***strong TMP*** *and positive entropy, then $\Gamma \curvearrowright X$ admits off-diagonal asymptotic pairs in the support of an invariant measure.*
- *If $\Gamma \curvearrowright X$ has* ***TMP*** *and off-diagonal asymptotic pairs in the support of an invariant measure, then $\Gamma \curvearrowright X$ has positive entropy. This one also works for sofic entropy!*

Thank you for your attention!

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