The domino problem for word-hyperbolic groups

Sebastián **Barbieri Lemp**
Joint work with Nathalie Aubrun and Etienne Moutot

University of British Columbia

Thirteenth International Conference on Computability, Complexity and Randomness

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The domino problem for **surface** groups

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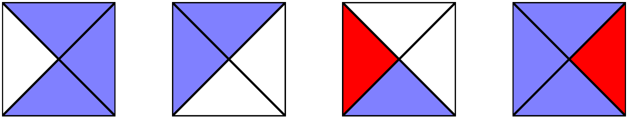
Classical domino problem

Consider a finite set τ of Wang tiles



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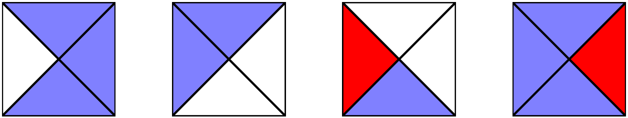
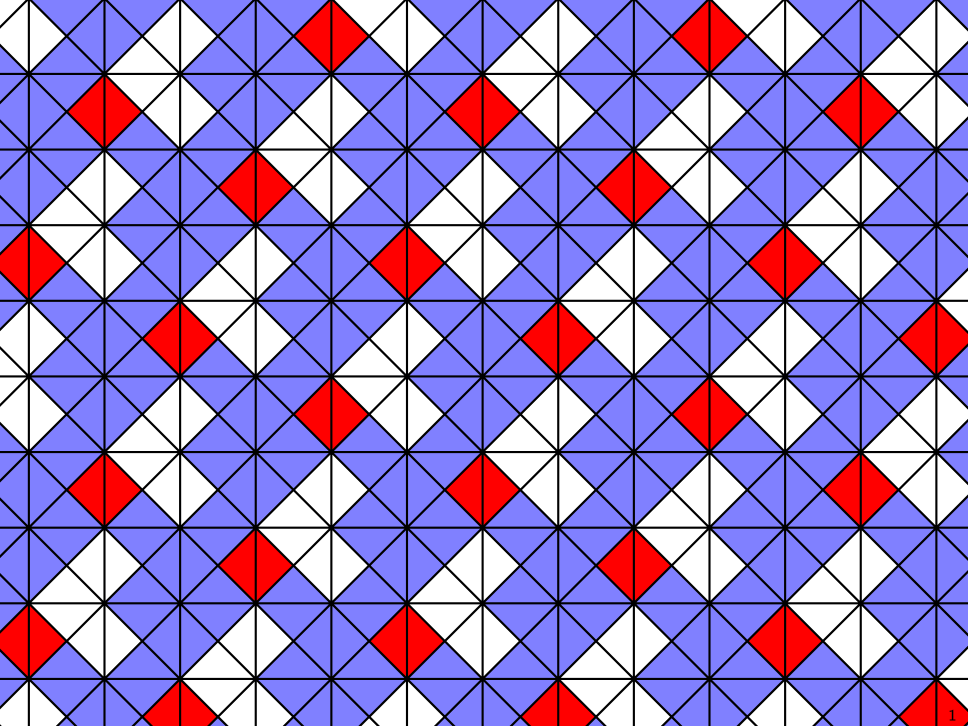


Question:

is there a function $x : \mathbb{Z}^2 \to \tau$ such that adjacent tiles share the same color?



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is there an algorithm which given a finite set of Wang tiles decides whether they tile the plane or not?

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Theorem (Berger 66')

No.

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The domino problem is `Undecidable`.

Domino problem

Soon, humankind started to explore new worlds...

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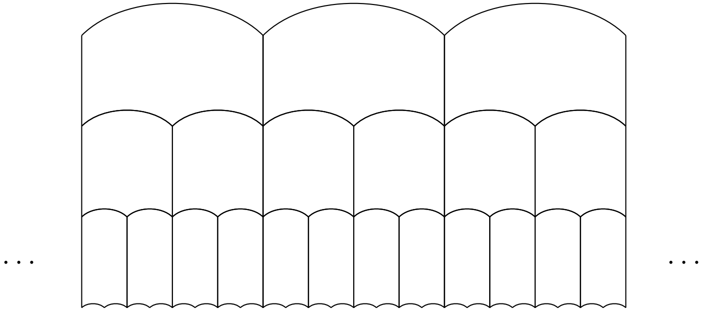

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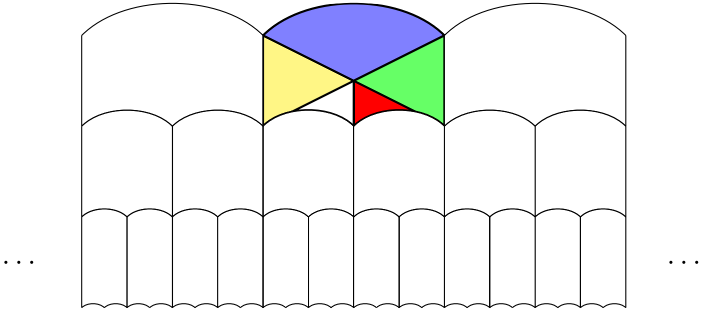

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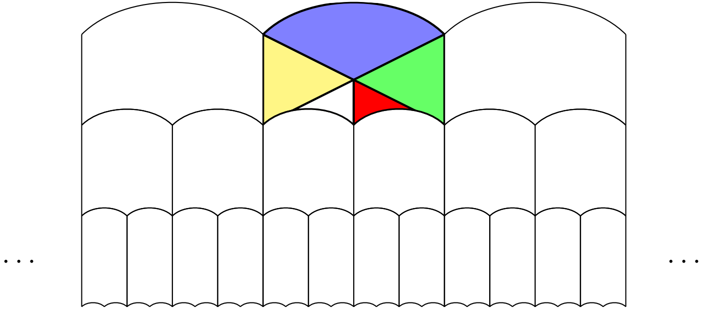

Theorem (Kari 08')

The domino problem is undecidable in the binary hyperbolic tiling.

General setting

Let us consider the following ingredients:

- A directed, labeled (infinite) graph $\Gamma = (V, E, L)$.
- A finite set of colors $\mathcal{A}$.
- A finite list of **forbidden** colored labeled connected finite graphs $\mathcal{F}$.

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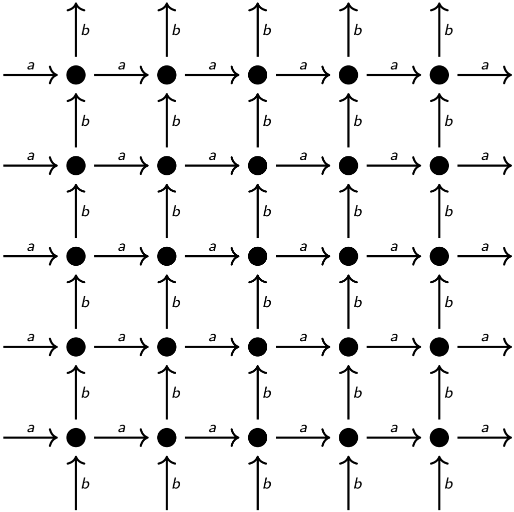
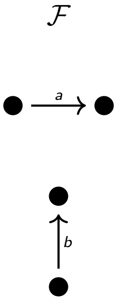
Domino problem for Γ:

Is there an algorithm which decides, given $(\mathcal{A}, \mathcal{F})$, whether there exists a coloring $x : V \to \mathcal{A}$ such that no graph from $\mathcal{F}$ embeds?

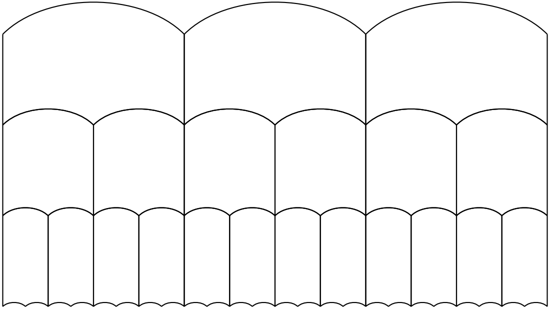
The original domino problem:





Binary hyperbolic tiling:

. . .



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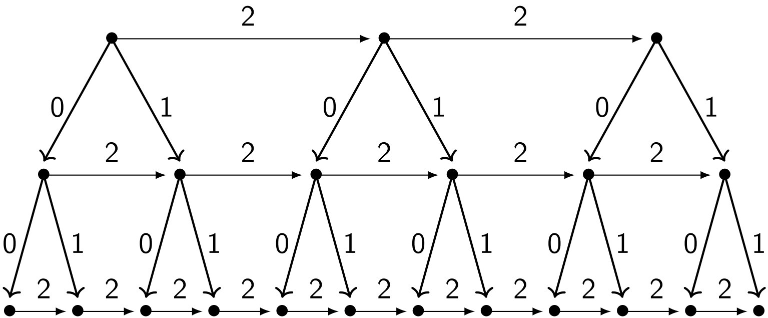

General setting: Cayley graphs

A particularly interesting case is when $\Gamma = (V, E, L)$ is the **Cayley graph** of a finitely generated group G given by the set of generators S.

- $V = G$.
- $E = \{(g, gs) \mid g \in G, s \in S\}$.
- $L(g, gs) = s$.

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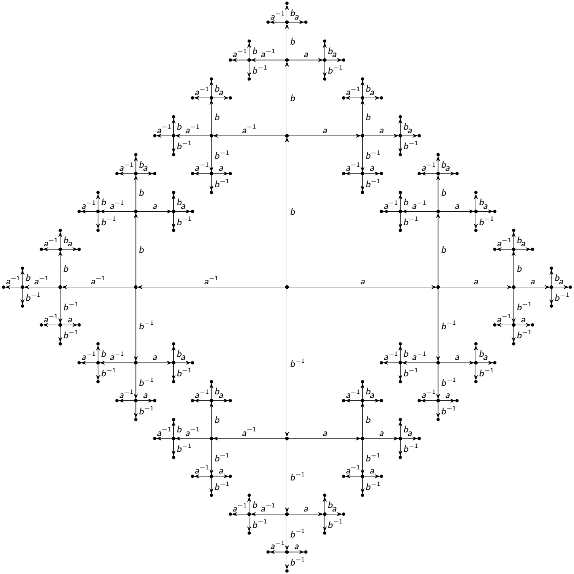
Remark: the domino problem does not depend upon the set of generators S. These problems are all computationally (many-one) equivalent.

$\texttt{DP}(G)$ is the domino problem of the group G.

Cayley graph of free group.



Domino problem on groups.

List of facts:

- $\mathtt{DP}(\mathbb{Z}^2)$ is undecidable.

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Domino conjecture

A finitely generated group has decidable domino problem if and only if it is virtually free.

Verified for polycyclic groups, Baumslag-Solitar groups, Branch groups.

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Theorem (Muller & Schupp '85)

A graph has decidable monadic second order logic (MSO) if and only if it has finite tree-width.

- **Fact 1**: A group is virtually free if and only if its Cayley graphs have finite tree-width.
- **Fact 2**: The domino problem can be expressed in MSO.

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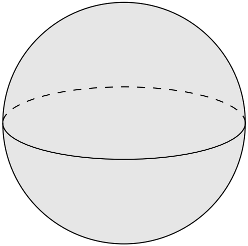
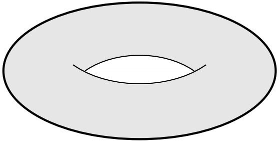
- **Fact 2**: The domino problem can be expressed in MSO.

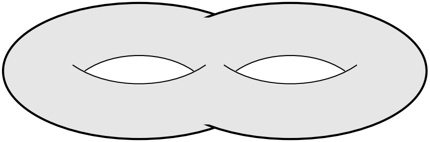
If DC holds, then the domino problem contains all the complexity of MSO for finitely generated groups.

Surface groups

Consider the fundamental group of a closed orientable surface.







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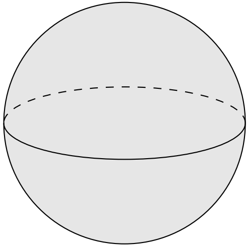

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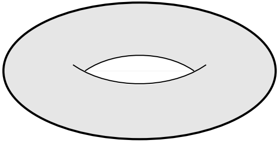

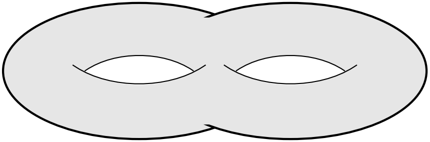

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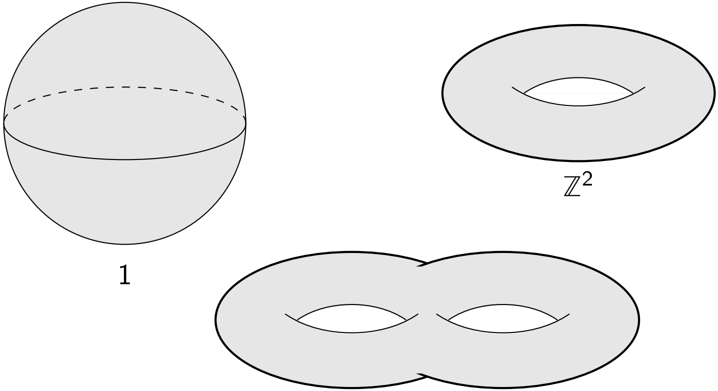

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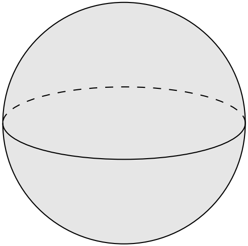

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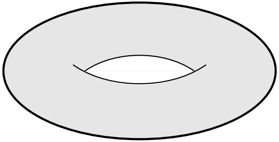
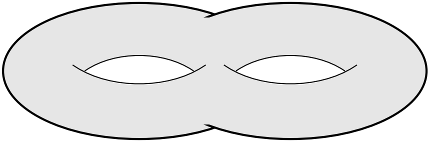
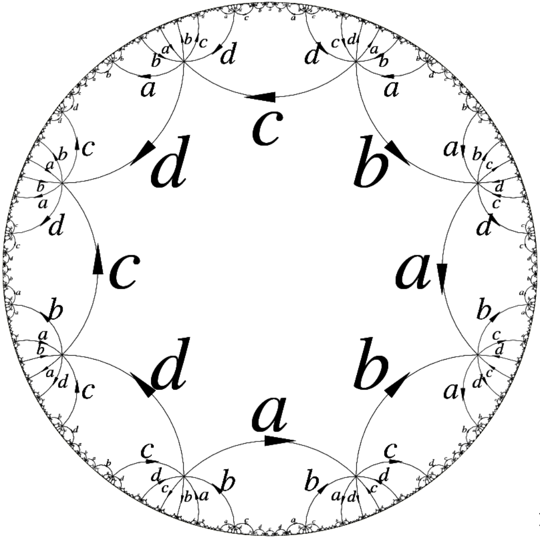


$\mathbb{Z}^2$



$\langle a, b, c, d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} = 1 \rangle$

Surface groups



[1]

[1] https://math.stackexchange.com/questions/1834108/cayley-graph-of-the-fundamental-group-of-the-2-torus

Surface groups

Theorem (Aubrun, B. Moutot)

The domino problem of the fundamental group of any closed orientable surface of positive genus is undecidable.

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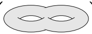
Remark: we just need to show that the domino problem of

π_1 () $\cong \langle a, b, c, d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} = 1 \rangle$

is undecidable.

How to prove it

Proof idea: use hyperbolicity.

- Step 1: show undecidability of `DP` for a class of graphs which embed nicely in the hyperbolic plane.

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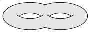
- Step 2: show that one of these graphs Γ can be "locally encoded" by a subshift of finite type (set of tilings given by forbidden patterns) of π_1 ().

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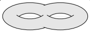
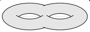
- Step 2: show that one of these graphs Γ can be "locally encoded" by a subshift of finite type (set of tilings given by forbidden patterns) of π_1 ().
- Step 3: reduce DP (π_1 ()) to DP(Γ).

How to prove it

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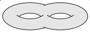
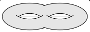
- Step 2: show that one of these graphs Γ can be "locally encoded" by a subshift of finite type (set of tilings given by forbidden patterns) of $\pi_1\Bigg($  $\Bigg)$.
- Step 3: reduce $\mathtt{DP}\Bigg(\pi_1\Bigg($  $\Bigg)\Bigg)$ to $\mathtt{DP}(\Gamma)$.
- Step 4: profit.

▸ Skip Proof

How to prove it: nice class

A (non-deterministic) **substitution** is a pair $(\mathcal{A}, R)$ where $\mathcal{A}$ is a finite alphabet and R is a set of pairs $(a \mapsto w) \in \mathcal{A} \times \mathcal{A}^*$.

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Example

- $\mathcal{A} = \{0\}, R = \{(0 \mapsto 00)\}$.
- $\mathcal{A} = \{0, 1\}, R = \{(1 \mapsto 0), (0 \mapsto 01)\}$.

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An infinite word $u = \dots u_{-1} u_0 u_1 u_2 \dots \in \mathcal{A}^{\mathbb{Z}}$ **produces** a word $v = \dots v_{-1} v_0 v_1 v_2 \dots \in \mathcal{A}^{\mathbb{Z}}$ if v can be obtained from u by applying a rule of R on each symbol.

That is, there exists a function $\Delta : \mathbb{Z} \to \mathbb{Z}$ such that :

$$(u_i \mapsto v_{\Delta(i)} \dots v_{\Delta(i+1)-1}) \in R \text{ for every } i \in \mathbb{Z}$$

How to prove it: nice class

Let $\{u_i\}_{i\in\mathbb{Z}}$ be a sequence of bi-infinite words such that u_i produces u_{i+1} (with Δ_i). We can associate an **orbit graph**.

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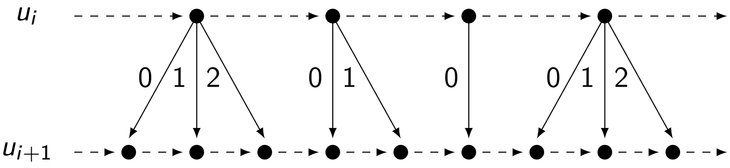
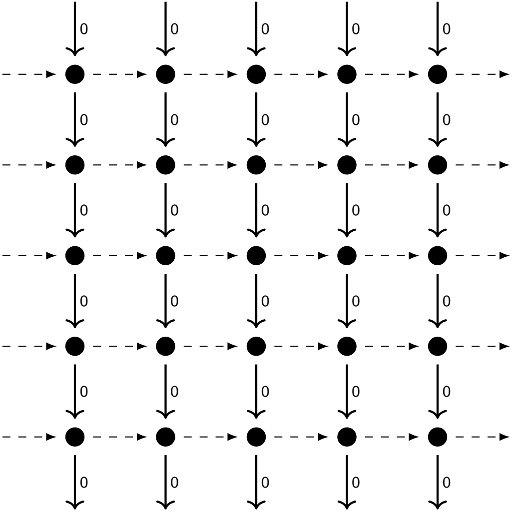


- Join all consecutive symbols of u_i by edges from left to right.
- Join each symbol of u_i with the corresponding sequence of symbols it produces in u_{i+1} assigning labels from left to right.

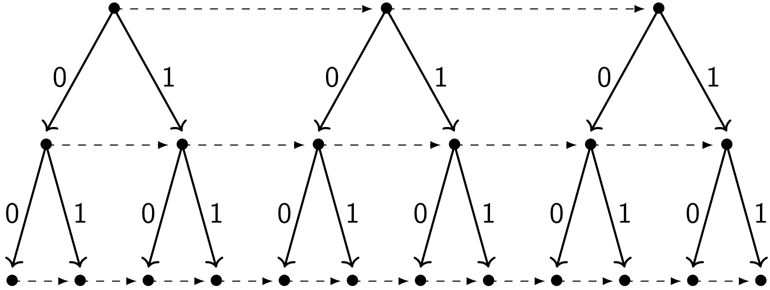
Example 1: trivial substitution gives $\mathbb{Z}^2$.

$\mathcal{A} = \{0\}$ $R = \{(0 \mapsto 0)\}$.



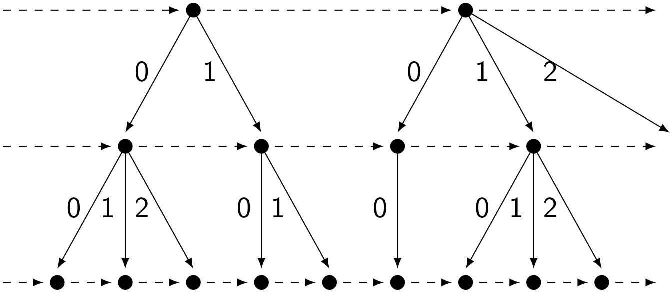
Example 2: Doubling substitution gives bin hyp tiling.

$\mathcal{A} = \{0\}$ $R = \{(0 \mapsto 00)\}$.



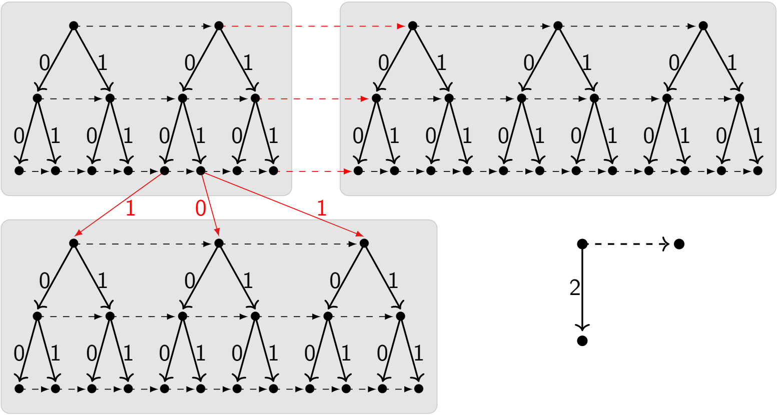
Undecidability: reduce to example 2.

Idea: take an orbit graph Γ.



Undecidability: reduce to example 2.

In each vertex code a finite subgraph of the binary orbit graph + information on how to locally paste them together.



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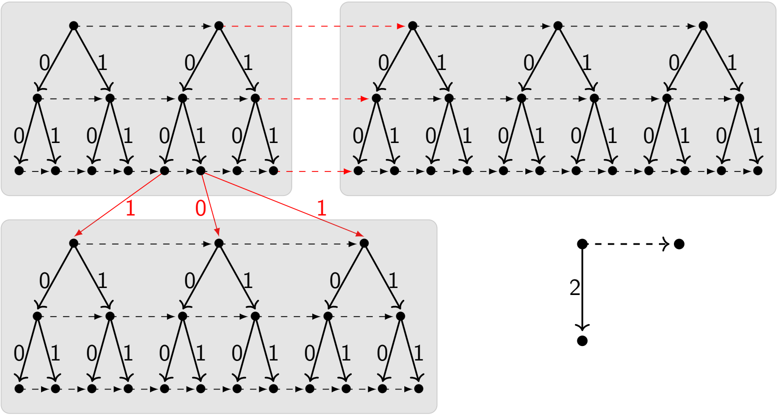

Impose local consistency rules.

Undecidability: reduce to example 2.

- Suppose $\mathtt{DP}(\Gamma)$ is decidable.
- Use the previous tiling to encode the binary orbit graph.
- Let $(\mathcal{A}, \mathcal{F})$ be an alphabet and a set of forbidden patterns for the binary orbit graph. Use the encoding to simulate tilings in Γ.
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- contradiction ✓.

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Warning

- We must check that the language of coded subgraphs is **finite**.
- We must check that the set of encodings is **non-empty**.

Hyperbolic geometry to the rescue!

A substitution $(\mathcal{A}, R)$ has an **expanding eigenvalue** if there exists $\lambda > 1$ and $v\colon \mathcal{A} \to \mathbb{R}^+$ such that for every$(a \mapsto u_1 \dots u_k) \in R$:

$$\lambda v(a) = (v(u_1) + v(u_2) + \dots + v(u_k))$$

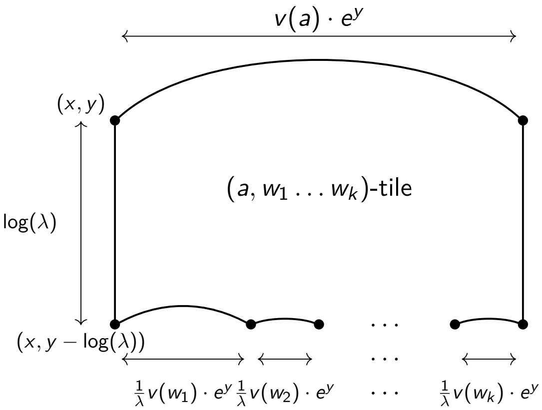
Example

$\mathcal{A} = \{0\}$ $R = \{(0 \mapsto 00)\}$ admits the expanding eigenvalue $\lambda = 2$.

$$2\lambda v(0) = (v(0) + v(0))$$

Hyperbolic geometry to the rescue!

To every orbit of a substitution with an expanding eigenvalue we can associate canonically a tiling of $\mathbb{H}^2$.



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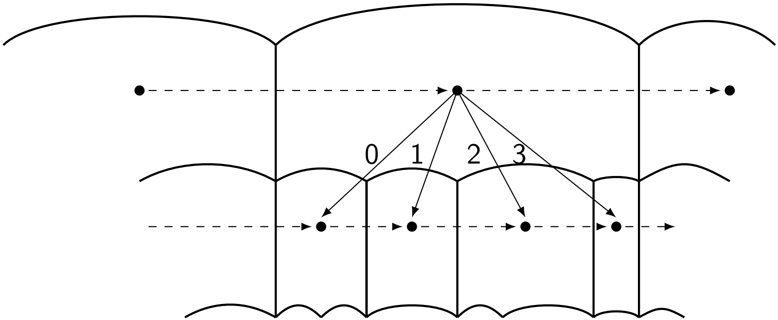

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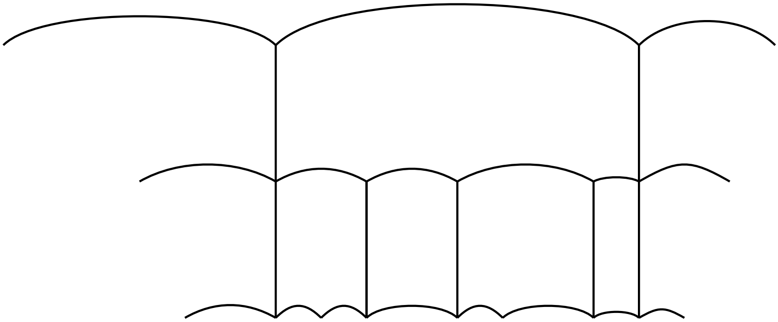

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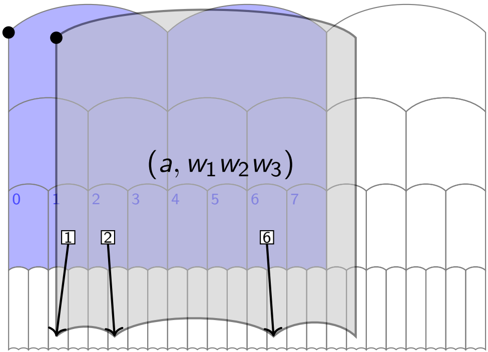
We superpose a tiling of $(\mathcal{A}, R)$ and a binary tiling.



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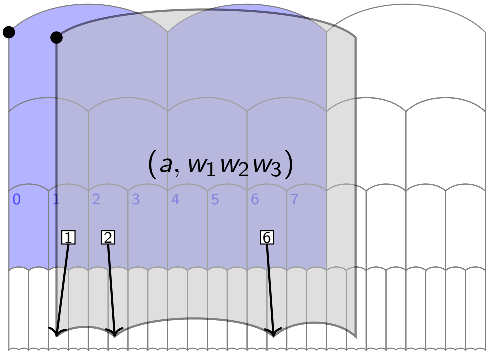

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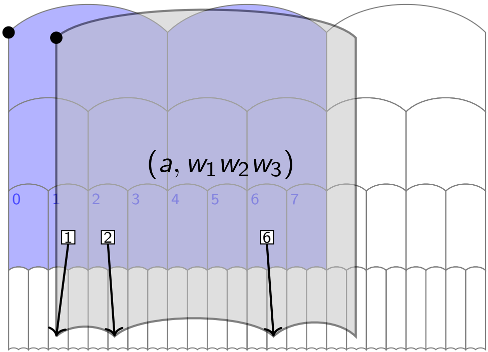
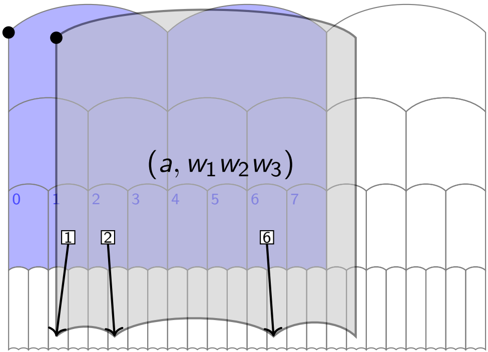


Finitely many (coded) ways to intersect $\Longrightarrow$ **finite alphabet**

There is an encoding $\Longrightarrow$ **non-emptiness**

Hyperbolic geometry to the rescue!

We superpose a tiling of $(\mathcal{A}, R)$ and a binary tiling.



Finitely many (coded) ways to intersect $\implies$ **finite alphabet**

There is an encoding $\implies$ **non-emptiness**

Remark: Tiling superpositions were introduced by D.B. Cohen and C. Goodman-Strauss to produce aperiodic tilings of surface groups.

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Theorem (Aubrun, B., Moutot)

For every orbit graph Γ *of a substitution with an expanding eigenvalue* $DP(\Gamma)$ *is undecidable.*

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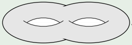
Question

How does this relate to the fundamental group of ?

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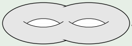
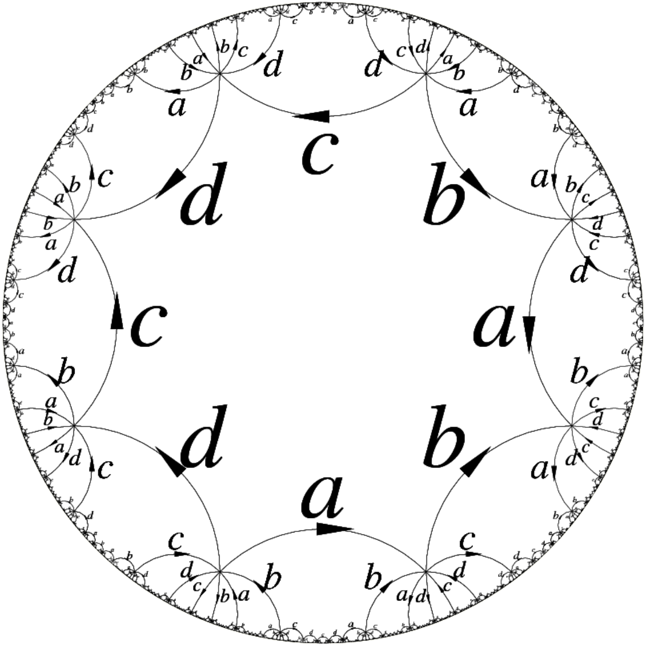
How does this relate to the fundamental group of ?

There is a "hidden" substitution in that group, namely $\mathcal{A} = \{\mathtt{a}, \mathtt{b}\}$ and

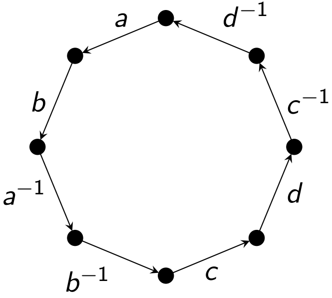
$$\left\{ \left(\mathtt{a} \mapsto \mathtt{ab}^5\mathtt{ab}^5\mathtt{ab}^5\mathtt{ab}^5\mathtt{ab}^4 \right), \left(\mathtt{b} \mapsto \mathtt{ab}^5\mathtt{ab}^5\mathtt{ab}^5\mathtt{ab}^5\mathtt{ab}^5\mathtt{ab}^4 \right). \right\}$$

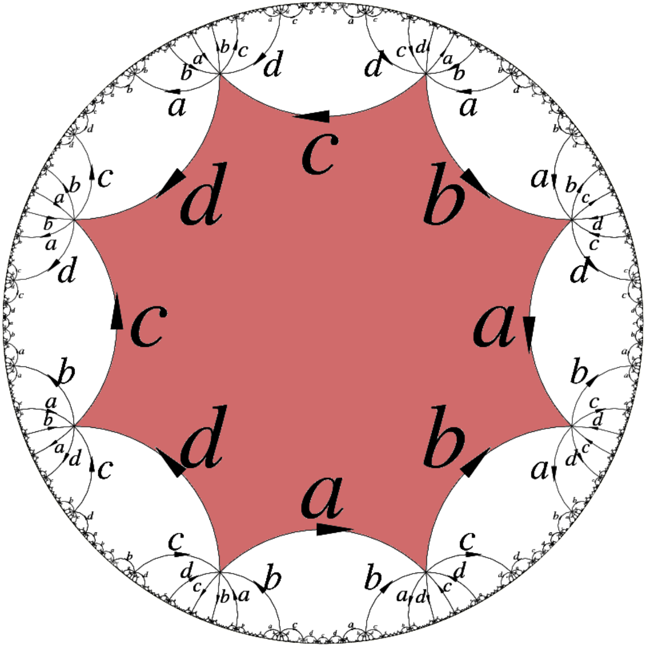
with $\lambda = 17 + 12\sqrt{2}$ and $v(\mathtt{b})/v(\mathtt{a}) = \frac{1+\sqrt{2}}{2}$.

▸ Skip Proof


c
d
b
c
a
d
b
a

A way to look at this Cayley graph is as a translation surface obtained by pasting together octagons.




c
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b
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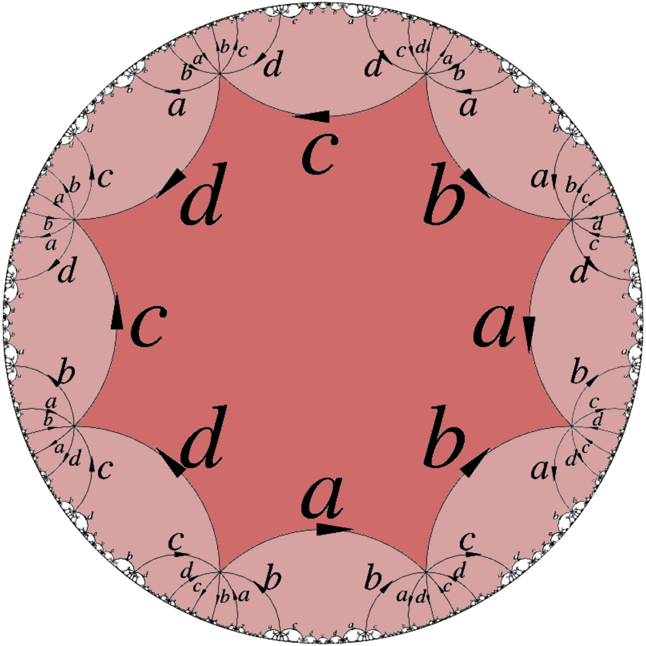
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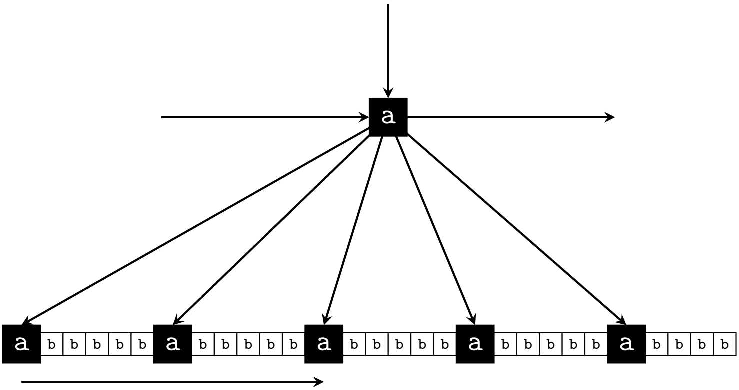
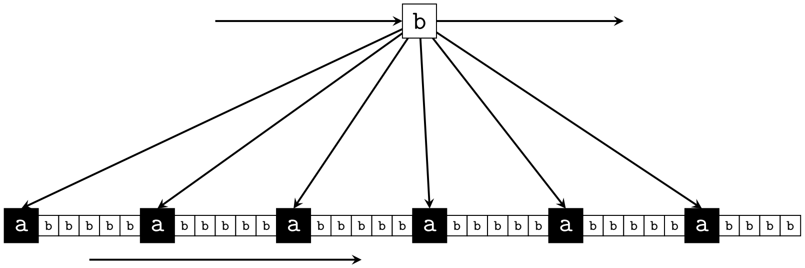
Surface group: vertex with ancestor

If a vertex is **connected** by a generator with the previous ring then the sequences of vertices in the next level it sees follows the following pattern:



Surface group: vertex with ancestor

If a vertex is **not connected** by a generator with the previous ring then the sequences of vertices in the next level it sees follows the following pattern:



Surface group: proof of undecidability

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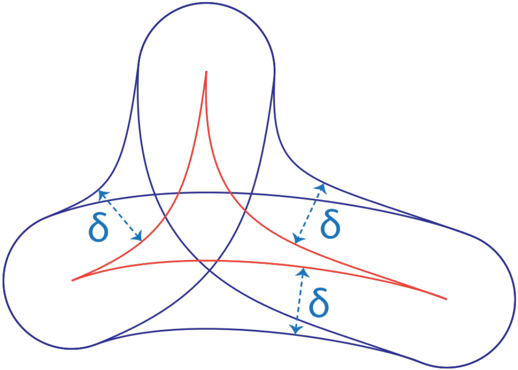
Theorem (Aubrun, B., Moutot)

The domino problem is undecidable on the fundamental group of the closed orientable surface of genus 2.

Word-hyperbolic groups

Word-hyperbolic group

A finitely generated group is **word-hyperbolic** if the geodesic triangles of one of its Cayley graphs are δ-slim for some $\delta > 0$.



2

[2] From https://en.wikipedia.org/wiki/Hyperbolic_group

Word-hyperbolic groups

Facts about word-hyperbolic groups:

- Virtually free groups ✓.
- Surface groups (genus $g \geq 2$) ✓.
- Nice computability properties: Finitely presented, decidable word problem, Dehn's algorithm works, language of shortlex geodesics is regular, etc.
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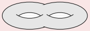
Bottom line: testing ground for

Domino conjecture

A finitely generated group has decidable domino problem if and only if it is virtually free.

Gromov's conjecture

The fundamental group of  embeds into any one-ended word-hyperbolic group.

word-hyperbolic groups

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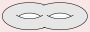
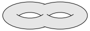
The fundamental group of  embeds into any one-ended word-hyperbolic group.

Facts:

- If a group H embeds into a group G, then the domino problem of G is computationally harder than the domino problem of H.
- If a word-hyperbolic group is not virtually free, it contains an embedded one-ended word-hyperbolic group.
- If GC holds, then every word-hyperbolic group which is not virtually free contains an embedded copy of the fundamental group of 

word-hyperbolic groups

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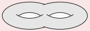
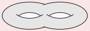
The fundamental group of  embeds into any one-ended word-hyperbolic group.

Theorem

If GC holds, then the domino problem conjecture holds for every word-hyperbolic group.

Gromov's conjecture

The fundamental group of  embeds into any one-ended word-hyperbolic group.

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If GC holds, then the domino problem conjecture holds for every word-hyperbolic group.

- **Fun fact**: find a (non virt free) word-hyperbolic group with decidable domino problem and you ~~shall attain fame and glory~~ disprove Gromov's conjecture!

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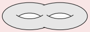
Gromov's conjecture

The fundamental group of  embeds into any one-ended word-hyperbolic group.

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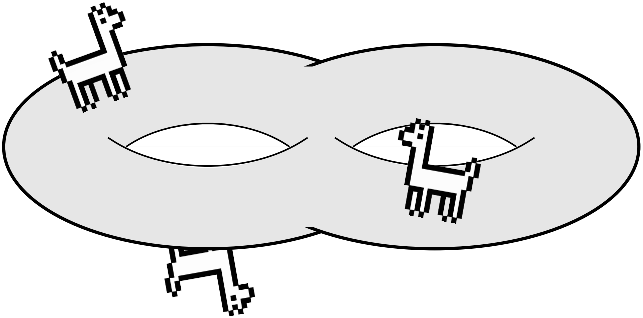
- **Fun fact**: Same can be shown with weaker versions of GC.

Thank you for your attention!



The domino problem is undecidable on surface groups.
https://arxiv.org/abs/1811.08420