A strongly aperiodic SFT in the Grigorchuk group.

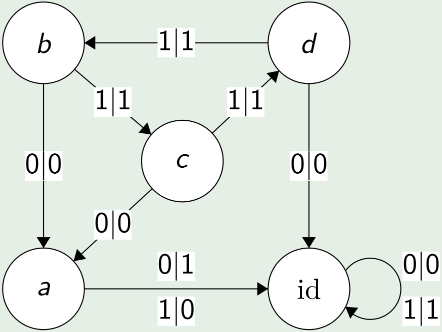
Sebastián **Barbieri Lemp**

(up to July 30th) LIP, ENS de Lyon – CNRS – INRIA – UCBL – Université de Lyon
(From August 1st) University of British Columbia

Pingree Park
July, 2017

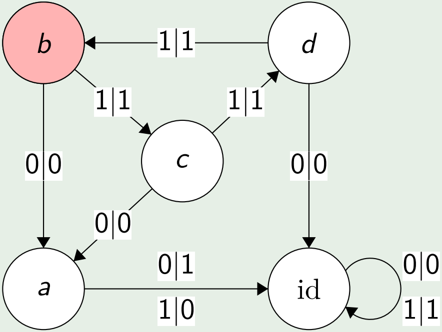
The Grigorchuk group

Generated by a, b, c, d acting over $\{0,1\}^{\mathbb{N}}$.



The Grigorchuk group

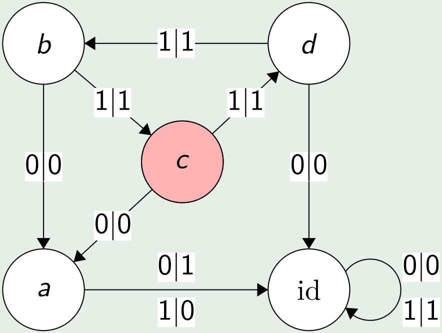
Generated by a, b, c, d acting over $\{0,1\}^{\mathbb{N}}$.



$$\begin{array}{cccccccccc} x = & 1 & 1 & 1 & 0 & 1 & 0 & 0 & \dots \\ \downarrow & b & & & & & & & \\ b(x) = & & & & & & & & \dots \end{array}$$

The Grigorchuk group

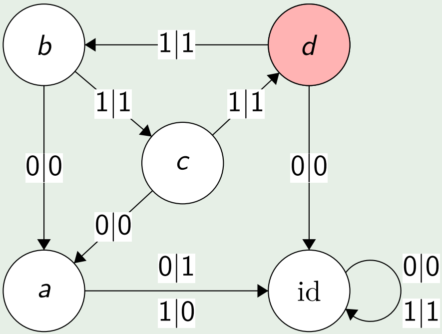
Generated by a, b, c, d acting over $\{0,1\}^{\mathbb{N}}$.



$$\begin{array}{rcccccccc} x = & 1 & 1 & 1 & 0 & 1 & 0 & 0 & \dots \\ \downarrow & b & c & & & & & & \\ b(x) = & 1 & & & & & & & \dots \end{array}$$

The Grigorchuk group

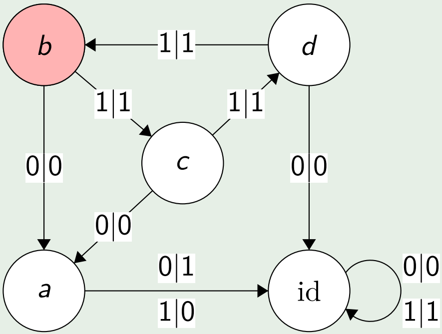
Generated by a, b, c, d acting over $\{0,1\}^{\mathbb{N}}$.



$$
\begin{array}{cccccccccc}
x = & 1 & 1 & 1 & 0 & 1 & 0 & 0 & \ldots \\
\downarrow & b & c & d & & & & & \\
b(x) = & 1 & 1 & & & & & & \ldots
\end{array}
$$

The Grigorchuk group

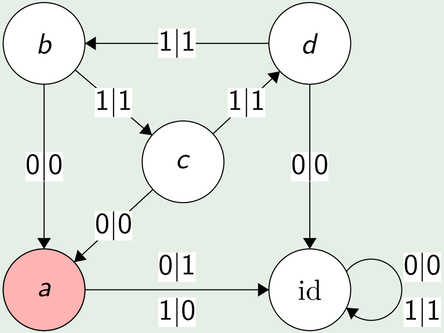
Generated by a, b, c, d acting over $\{0,1\}^{\mathbb{N}}$.



$$\begin{array}{rccccccccc} x = & 1 & 1 & 1 & 0 & 1 & 0 & 0 & \dots \\ \downarrow & b & c & d & b & & & & \\ b(x) = & 1 & 1 & 1 & & & & & \dots \end{array}$$

The Grigorchuk group

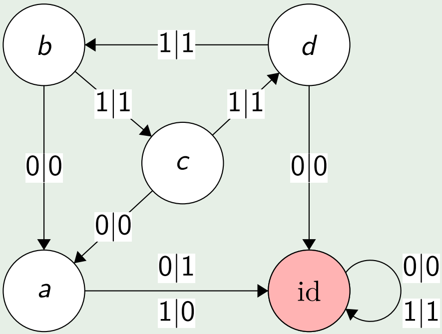
Generated by a, b, c, d acting over $\{0,1\}^{\mathbb{N}}$.



$$\begin{array}{cccccccccc} x = & 1 & 1 & 1 & 0 & 1 & 0 & 0 & \dots \\ \downarrow & b & c & d & b & a & & & \\ b(x) = & 1 & 1 & 1 & 0 & & & & \dots \end{array}$$

The Grigorchuk group

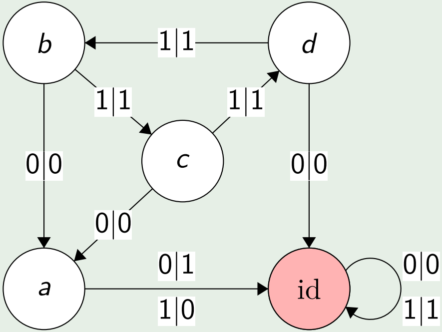
Generated by a, b, c, d acting over $\{0,1\}^{\mathbb{N}}$.



$$
\begin{array}{lccccccccc}
x = & 1 & 1 & 1 & 0 & 1 & 0 & 0 & \dots \\
\downarrow & b & c & d & b & a & \mathrm{id} & & \\
b(x) = & 1 & 1 & 1 & 0 & 0 & & & \dots
\end{array}
$$

The Grigorchuk group

Generated by a, b, c, d acting over $\{0,1\}^{\mathbb{N}}$.



$$\begin{array}{ccccccccc} x = & 1 & 1 & 1 & 0 & 1 & 0 & 0 & \dots \\ \downarrow & b & c & d & b & a & \mathrm{id} & \mathrm{id} & \\ b(x) = & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \dots \end{array}$$

What about the Grigorchuk group?

- a, b, c, d are involutions.
- The Grigorchuk group is infinite and finitely generated.
- It contains no copy of $\mathbb{Z}$ as a subgroup. For every $g \in G$, there is $n \in \mathbb{N}$ such that $g^n = 1_G$.
- Decidable word problem (and conjugacy problem).
- It has intermediate growth.
- It is commensurable to its square. ie: G and $G \times G$ have an isomorphic finite index subgroup.

What about the Grigorchuk group?

- a, b, c, d are involutions.
- The Grigorchuk group is infinite and finitely generated.
- It contains no copy of $\mathbb{Z}$ as a subgroup. For every $g \in G$, there is $n \in \mathbb{N}$ such that $g^n = 1_G$.
- Decidable word problem (and conjugacy problem).
- It has intermediate growth.
- It is commensurable to its square. ie: G and $G \times G$ have an isomorphic finite index subgroup.

The goal of this talk is to construct a strongly aperiodic SFT here.

Definitions

- G is a finitely generated group.
- $\mathcal{A}$ is a finite alphabet. Ex: $\mathcal{A} = \{0, 1\}$.
- $\mathcal{A}^G$ is the set of configurations, $x : G \to \mathcal{A}$
- $\sigma : G \times \mathcal{A}^G \to \mathcal{A}^G$ is the left shift action given by:

$$\sigma(h, x)_g := \sigma^h(x)_g = x_{h^{-1}g}.$$

Definitions

- G is a finitely generated group.
- $\mathcal{A}$ is a finite alphabet. Ex: $\mathcal{A} = \{0, 1\}$.
- $\mathcal{A}^G$ is the set of configurations, $x : G \to \mathcal{A}$
- $\sigma : G \times \mathcal{A}^G \to \mathcal{A}^G$ is the left shift action given by:

$$\sigma(h, x)_g := \sigma^h(x)_g = x_{h^{-1}g}.$$

Definition: subshift

The pair (X, σ) where $X \subset \mathcal{A}^G$ is a closed and shift-invariant set is called a *subshift*.

Definitions

- G is a finitely generated group.
- $\mathcal{A}$ is a finite alphabet. Ex: $\mathcal{A} = \{0, 1\}$.
- $\mathcal{A}^G$ is the set of configurations, $x : G \to \mathcal{A}$
- $\sigma : G \times \mathcal{A}^G \to \mathcal{A}^G$ is the left shift action given by:

$$\sigma(h, x)_g := \sigma^h(x)_g = x_{h^{-1}g}.$$

Definition: subshift

The pair (X, σ) where $X \subset \mathcal{A}^G$ is a closed and shift-invariant set is called a *subshift*.

A subshift is a set of configurations avoiding patterns from a list $\mathcal{F}$.

$$p \in \mathcal{A}^S, \qquad [p] = \{x \in \mathcal{A}^G \mid x|_S = p\}$$

$$X = X_{\mathcal{F}} = \mathcal{A}^G \setminus \bigcup_{g \in G, p \in \mathcal{F}} \sigma^g([p])$$

Definitions

Classes of subshifts

A subshift $X \subset \mathcal{A}^G$ is called:

- a *subshift of finite type (SFT)* if $X = X_{\mathcal{F}}$ for some finite $\mathcal{F}$.

Definitions

Classes of subshifts

A subshift $X \subset \mathcal{A}^G$ is called:

- a *subshift of finite type (SFT)* if $X = X_{\mathcal{F}}$ for some finite $\mathcal{F}$.
- a *sofic subshift* if X is the image of an SFT by a topological factor (a local recoding).

Definitions

Classes of subshifts

A subshift $X \subset \mathcal{A}^G$ is called:

- a *subshift of finite type (SFT)* if $X = X_{\mathcal{F}}$ for some finite $\mathcal{F}$.
- a *sofic subshift* if X is the image of an SFT by a topological factor (a local recoding).
- an *effectively closed subshift* if X can be defined by a recursively enumerable coding of a set of forbidden patterns.

Definitions

Classes of subshifts

A subshift $X \subset \mathcal{A}^G$ is called:

- a *subshift of finite type (SFT)* if $X = X_{\mathcal{F}}$ for some finite $\mathcal{F}$.
- a *sofic subshift* if X is the image of an SFT by a topological factor (a local recoding).
- an *effectively closed subshift* if X can be defined by a recursively enumerable coding of a set of forbidden patterns.

Strongly aperiodic

A subshift $X \subset \mathcal{A}^G$ is *strongly aperiodic* if the shift action is free.

$$\forall x \in X, \sigma^g(x) = x \implies g = 1_G.$$

Problem

Question

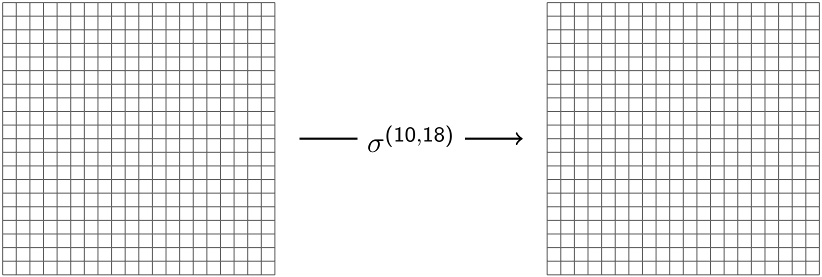
Which groups admit strongly aperiodic SFTs?

Problem

Question

Which groups admit strongly aperiodic SFTs?

Baby (alpaca) example: Let $G = \mathbb{Z}^2/20\mathbb{Z}^2$

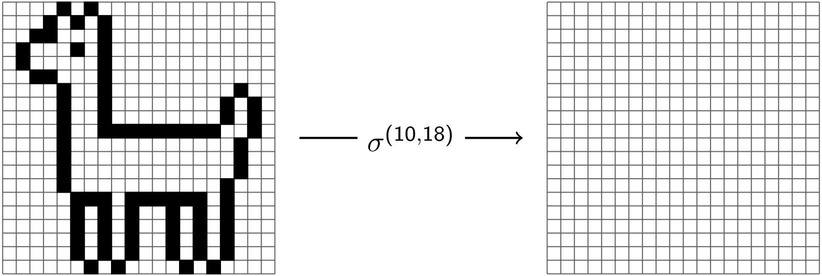


Problem

Question

Which groups admit strongly aperiodic SFTs?

Baby (alpaca) example: Let $G = \mathbb{Z}^2/20\mathbb{Z}^2$

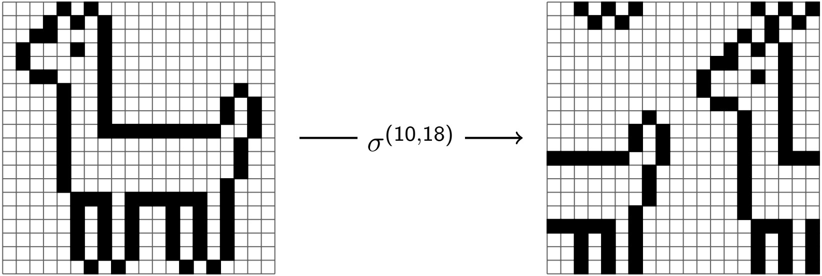


Problem

Question

Which groups admit strongly aperiodic SFTs?

Baby (alpaca) example: Let $G = \mathbb{Z}^2/20\mathbb{Z}^2$



An application: strongly aperiodic subshifts

Proposition

Every non-empty $\mathbb{Z}$-SFT contains a periodic configuration.

An application: strongly aperiodic subshifts

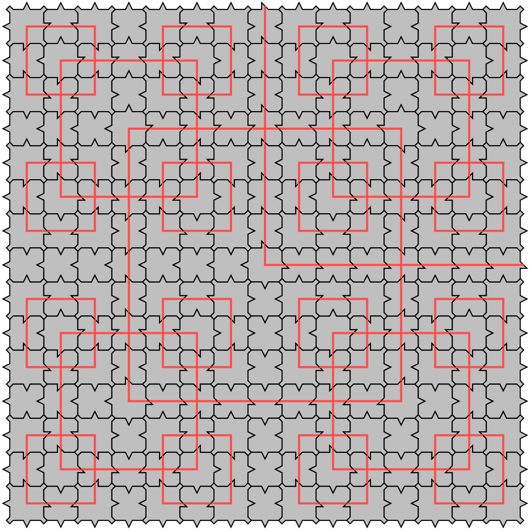
Proposition

Every non-empty $\mathbb{Z}$-SFT contains a periodic configuration.

Theorem (Berger 1966, Robinson 1971, Kari 1996, Jeandel & Rao 2015)

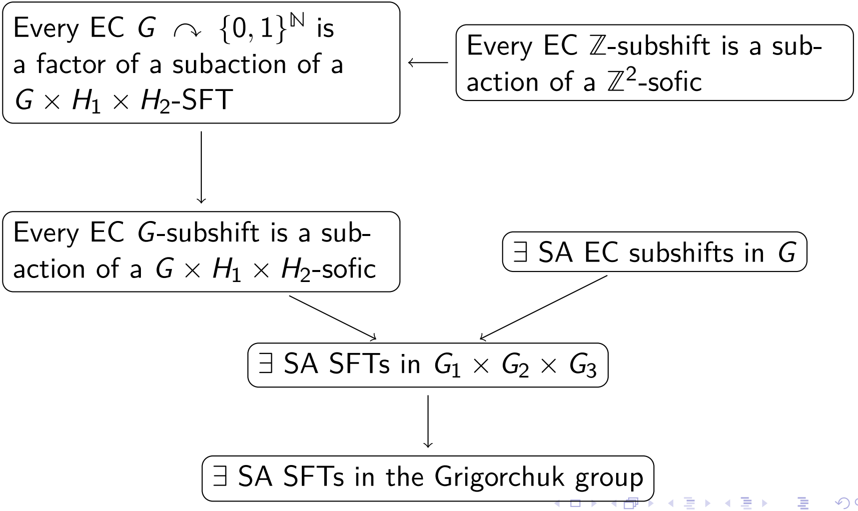
There exist strongly aperiodic SFTs on $\mathbb{Z}^2$.

Example of strongly aperiodic $\mathbb{Z}^2$-SFT: Robinson tileset



What about the Grigorchuk group?

All groups here are infinite, finitely generated and have decidable word problem.



What about the Grigorchuk group?

All groups here are infinite, finitely generated and have decidable word problem.

Every EC $G \curvearrowright \{0,1\}^{\mathbb{N}}$ is a factor of a subaction of a $G \times H_1 \times H_2$-SFT

Every EC $\mathbb{Z}$-subshift is a subaction of a $\mathbb{Z}^2$-sofic

AS 2010, DRS 2010

Every EC G-subshift is a subaction of a $G \times H_1 \times H_2$-sofic

$\exists$ SA EC subshifts in G

ABT 2017+

$\exists$ SA SFTs in $G_1 \times G_2 \times G_3$

$\exists$ SA SFTs in the Grigorchuk group

Reverse chronology

Commensurability

We say that two groups G_1, G_2 are *commensurable* if they contain finite index subgroups H_1, H_2 such that $H_1 \cong H_2$.

$$G_1 \hookleftarrow H_1 \cong H_2 \hookrightarrow G_2$$

Reverse chronology

Commensurability

We say that two groups G_1, G_2 are *commensurable* if they contain finite index subgroups H_1, H_2 such that $H_1 \cong H_2$.

$$G_1 \hookleftarrow H_1 \cong H_2 \hookrightarrow G_2$$

▷ Recall that the Grigorchuk group G is commensurable to its square $G \times G$

Reverse chronology

Commensurability

We say that two groups G_1, G_2 are *commensurable* if they contain finite index subgroups H_1, H_2 such that $H_1 \cong H_2$.

$$G_1 \hookleftarrow H_1 \cong H_2 \hookrightarrow G_2$$

▷ Recall that the Grigorchuk group G is commensurable to its square $G \times G$
▷ if G is commensurable to $G \times G$, then it is also commensurable to $G \times G \times G$.

Reverse chronology

Commensurability

We say that two groups G_1, G_2 are *commensurable* if they contain finite index subgroups H_1, H_2 such that $H_1 \cong H_2$.

$$G_1 \hookleftarrow H_1 \cong H_2 \hookrightarrow G_2$$

▷ Recall that the Grigorchuk group G is commensurable to its square $G \times G$
▷ if G is commensurable to $G \times G$, then it is also commensurable to $G \times G \times G$.

Theorem (Carroll-Penland, 2015)

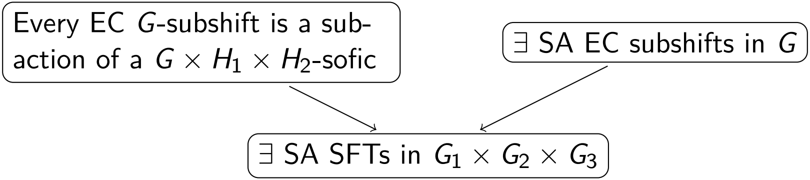
Admitting a strongly aperiodic SFT is a commensurability invariant.

$\exists$ SA SFTs in $G_1 \times G_2 \times G_3$

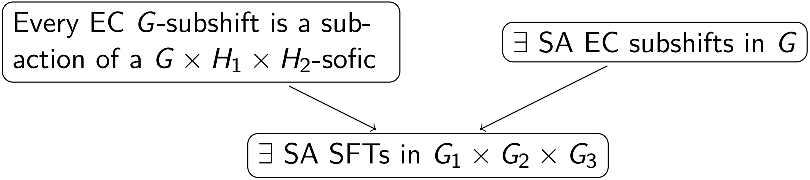
$\downarrow$

$\exists$ SA SFTs in the Grigorchuk group

We want to show next:



We want to show next:



First a little bit of philosophy.

The philosophy behind it

Finitely presented group

A group G is finitely presented if it can be described as $G = \langle S | R \rangle$ where both S and $R \subset (S \cup S^{-1})^*$ are finite.

$$\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle$$

The philosophy behind it

Finitely presented group

A group G is finitely presented if it can be described as $G = \langle S | R \rangle$ where both S and $R \subset (S \cup S^{-1})^*$ are finite.

$$\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle$$

Recursively presented group

A group G is recursively presented if it can be described as $G = \langle S | R \rangle$ where $S \subset \mathbb{N}$ and $R \subset (S \cup S^{-1})^*$ are recursive sets.

$$L = \langle a, t \mid (at^n a t^{-n})^2, n \in \mathbb{N} \rangle$$

The philosophy behind it

Finitely presented group

A group G is finitely presented if it can be described as $G = \langle S|R \rangle$ where both S and $R \subset (S \cup S^{-1})^*$ are finite.

$$\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle$$

Recursively presented group

A group G is recursively presented if it can be described as $G = \langle S|R \rangle$ where $S \subset \mathbb{N}$ and $R \subset (S \cup S^{-1})^*$ are recursive sets.

$$L = \langle a, t \mid (at^n at^{-n})^2, n \in \mathbb{N} \rangle$$

$$\bigoplus_{i \in \mathbb{N}} \mathbb{Z}/2\mathbb{Z} \cong \langle a_n, n \in \mathbb{N} \mid \{a_n^2\}_{n \in \mathbb{N}}, [a_j, a_k]_{j,k \in \mathbb{N}} \rangle.$$

The philosophy behind it

Theorem (Highman 1961)

For every recursively presented group H there exists a finitely presented group G such that H is isomorphic to a subgroup of G.

The philosophy behind it

Theorem (Highman 1961)

For every recursively presented group H there exists a finitely presented group G such that H is isomorphic to a subgroup of G.

"A complicated object is realized inside another object which admits a much simpler presentation."

The philosophy behind it

Theorem (Highman 1961)

For every recursively presented group H there exists a finitely presented group G such that H is isomorphic to a subgroup of G.

"A complicated object is realized inside another object which admits a much simpler presentation."

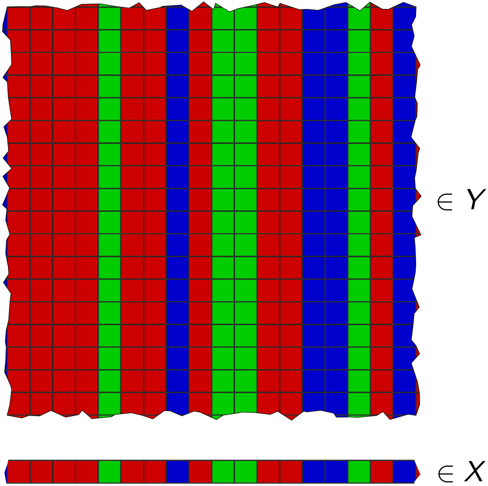
Corollary [Theorem: Novikov 1955, Boone 1958]

There are finitely presented groups with undecidable word problem

Just apply Highman's theorem to
$G = \langle a, b, c, d \mid b^{-n}ab^{n} = c^{-n}dc^{n}, n \in \texttt{HALT} \rangle$... done!

The case of subshifts

Every EC $\mathbb{Z}$-subshift X is a subaction of a $\mathbb{Z}^2$-sofic Y



So... why is simulation important?

It is complicated to come up with $\mathbb{Z}^2$-SFTs which are strongly aperiodic, however, finding a $\mathbb{Z}$-effectively closed subshift which is aperiodic is easy.

So... why is simulation important?

It is complicated to come up with $\mathbb{Z}^2$-SFTs which are strongly aperiodic, however, finding a $\mathbb{Z}$-effectively closed subshift which is aperiodic is easy.

Example

Let x be a fixed point of the Thue-Morse substitution.

$$0 \to 01 \to 0110 \to 01101001 \to 0110100110010110 \to \dots$$

Then $X = \overline{\mathrm{Orb}_\sigma(x)}$ is strongly aperiodic and effectively closed.

So... why is simulation important?

It is complicated to come up with $\mathbb{Z}^2$-SFTs which are strongly aperiodic, however, finding a $\mathbb{Z}$-effectively closed subshift which is aperiodic is easy.

Example

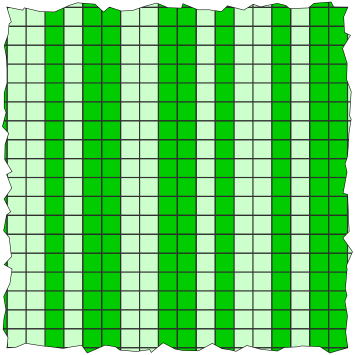
Let x be a fixed point of the Thue-Morse substitution.

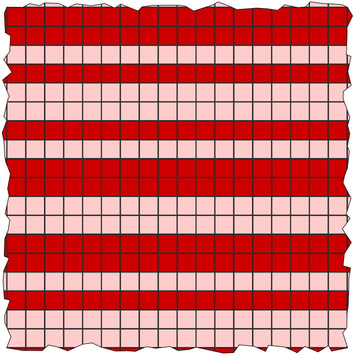
$$0 \to 01 \to 0110 \to 01101001 \to 0110100110010110 \to \dots$$

Then $X = \overline{\mathrm{Orb}_\sigma(x)}$ is strongly aperiodic and effectively closed.

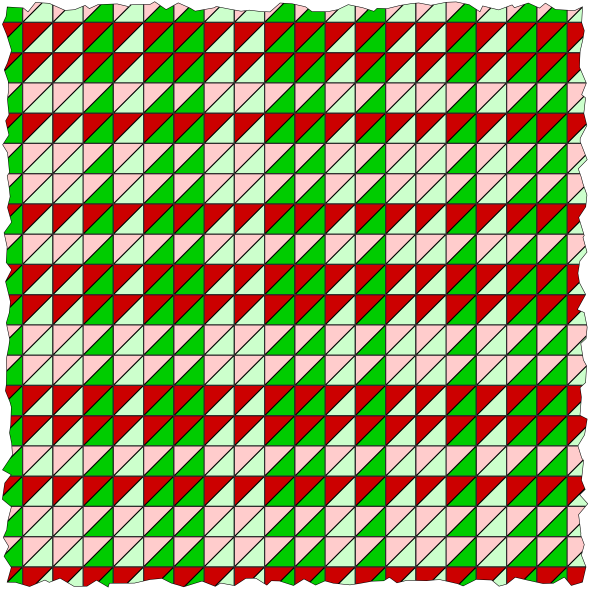
Example

A Sturmian subshift given by a computable slope α.





So... why is simultation important?



In our case

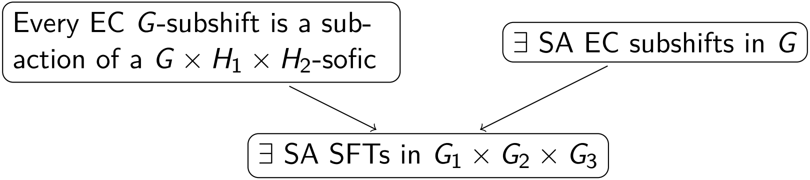
proof

- Take G_1 EC SA subshift. Use simulation to obtain a $G_1 \times G_2 \times G_3$-sofic subshift Y_1 such that σ acts trivially under $G_2 \times G_3$ and freely under G_1.
- Do the same for G_2, G_3 to get Y_2, Y_3.
- $Y_1 \times Y_2 \times Y_3$ is a SA sofic subshift.
- Any SFT extension $X \twoheadrightarrow Y_1 \times Y_2 \times Y_3$ works.

In our case

proof

- Take G_1 EC SA subshift. Use simulation to obtain a $G_1 \times G_2 \times G_3$-sofic subshift Y_1 such that σ acts trivially under $G_2 \times G_3$ and freely under G_1.
- Do the same for G_2, G_3 to get Y_2, Y_3.
- $Y_1 \times Y_2 \times Y_3$ is a SA sofic subshift.
- Any SFT extension $X \twoheadrightarrow Y_1 \times Y_2 \times Y_3$ works.



How does one prove such a thing?

Let's keep it simple, let's do $G \times \mathbb{Z}^2$. Consider an action

$G \curvearrowright X \subset \{0,1\}^{\mathbb{N}}$ (not necessarily expansive).

How does one prove such a thing?

Let's keep it simple, let's do $G \times \mathbb{Z}^2$. Consider an action

$G \curvearrowright X \subset \{0,1\}^{\mathbb{N}}$ (not necessarily expansive). Let

$\Psi : \{0,1\}^{\mathbb{N}} \to \{0,1,\$\}^{\mathbb{Z}}$ be given by:

$$\Psi(x)_j = \begin{cases} x_n & \text{if } j = 3^n \mod 3^{n+1} \\ \$ & \text{in the contrary case.} \end{cases}$$

How does one prove such a thing?

Let's keep it simple, let's do $G \times \mathbb{Z}^2$. Consider an action

$G \curvearrowright X \subset \{0,1\}^{\mathbb{N}}$ (not necessarily expansive). Let

$\Psi : \{0,1\}^{\mathbb{N}} \to \{0,1,\$\}^{\mathbb{Z}}$ be given by:

$$\Psi(x)_j = \begin{cases} x_n & \text{if } j = 3^n \mod 3^{n+1} \\ \$ & \text{in the contrary case.} \end{cases}$$

Example

If we write $x = x_0x_1x_2x_3\dots$ we obtain,

$$\Psi(x) = \dots\$x_0\$x_1x_0\$\$x_0\$x_2x_0\$x_1x_0\$\$x_0\$\$x_0\$x_1x_0\$\$x_0\$x_3x_0\dots$$

How does one prove such a thing?

$$\ldots \$x_0\$x_1x_0\$\$x_0\$x_2x_0\$x_1x_0\$\$x_0\$\$x_0\$x_1x_0\$\$x_0\$x_3x_0\$ \ldots$$

How does one prove such a thing?

$$\ldots \$x_0\$x_1x_0\$\$x_0\$x_2x_0\$x_1x_0\$\$x_0\$\$x_0\$x_1x_0\$\$x_0\$x_3x_0\$ \ldots$$

$$\downarrow$$

$$\ldots \$\color{red}{x_0\$}x_1\color{red}{x_0\$}\$\color{red}{x_0\$}x_2\color{red}{x_0\$}x_1\color{red}{x_0\$}\$\color{red}{x_0\$}\$\color{red}{x_0\$}x_1\color{red}{x_0\$}\$\color{red}{x_0\$}x_3\color{red}{x_0\$} \ldots$$

How does one prove such a thing?

$$\ldots \$x_0\$x_1x_0\$\$x_0\$x_2x_0\$x_1x_0\$\$x_0\$\$x_0\$x_1x_0\$\$x_0\$x_3x_0\$ \ldots$$

$$\downarrow$$

$$\ldots \$x_0\$x_1x_0\$\$x_0\$x_2x_0\$x_1x_0\$\$x_0\$\$x_0\$x_1x_0\$\$x_0\$x_3x_0\$ \ldots$$

$$\downarrow$$

$$\ldots \$ \quad x_1 \quad \$ \quad x_2 \quad x_1 \quad \$ \quad \$ \quad x_1 \quad \$ \quad x_3 \quad \ldots$$

How does one prove such a thing?

$$\ldots \$x_0\$x_1x_0\$\$x_0\$x_2x_0\$x_1x_0\$\$x_0\$\$x_0\$x_1x_0\$\$x_0\$x_3x_0\$ \ldots$$

$$\downarrow$$

$$\ldots \$x_0\$x_1x_0\$\$x_0\$x_2x_0\$x_1x_0\$\$x_0\$\$x_0\$x_1x_0\$\$x_0\$x_3x_0\$ \ldots$$

$$\downarrow$$

$$\ldots \$ \quad x_1 \quad \$ \quad x_2 \quad x_1 \quad \$ \quad \$ \quad x_1 \quad \$ \quad x_3 \quad \ldots$$

$$\downarrow$$

$$\ldots \$x_1\$x_2x_1\$\$x_1\$x_3x_1\$x_2x_1\$\$x_1\$\$x_1\$x_2x_1\$\$x_1\$x_4x_1\$ \ldots$$

How does one prove such a thing?

▷ pick a finite set of generators S of G.

▷ construct a subshift Π where every configuration is an S-tuple of configurations of the previous form.

$$S = \{1_G, s_1, \dots s_n\}$$

$$(\Psi(x), \Psi(T^{s_1})(x), \dots, \Psi(T^{s_n}(x))) \in \Pi$$

How does one prove such a thing?

▷ pick a finite set of generators S of G.
▷ construct a subshift Π where every configuration is an S-tuple of configurations of the previous form.

$$S = \{1_G, s_1, \ldots s_n\}$$

$$(\Psi(x), \Psi(T^{s_1})(x), \ldots, \Psi(T^{s_n}(x))) \in \Pi$$

Claim

If T is an effectively closed action, Π is effectively closed.

How does one prove such a thing?

▷ Take Π and construct a sofic $\mathbb{Z}^2$ subshift $\widetilde{\Pi}$ having Π in every horizontal row using the expansive simulation theorem.

How does one prove such a thing?

▷ Take Π and construct a sofic $\mathbb{Z}^2$ subshift $\widetilde{\Pi}$ having Π in every horizontal row using the expansive simulation theorem.

▷ Using the decoding argument, construct a map from Π to X.

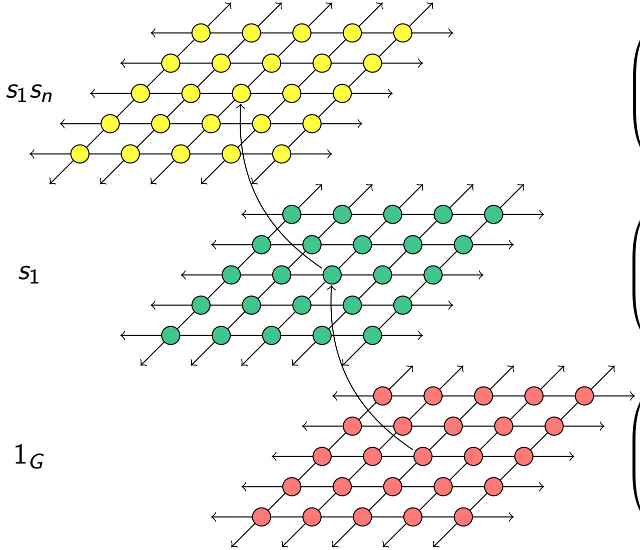
How does one prove such a thing?

▷ Take Π and construct a sofic $\mathbb{Z}^2$ subshift $\widetilde{\Pi}$ having Π in every horizontal row using the expansive simulation theorem.

▷ Using the decoding argument, construct a map from Π to X.

▷ Put in every G-coset of $G \times \mathbb{Z}^2$ a configuration of $\widetilde{\Pi}$.

How does one prove such a thing?

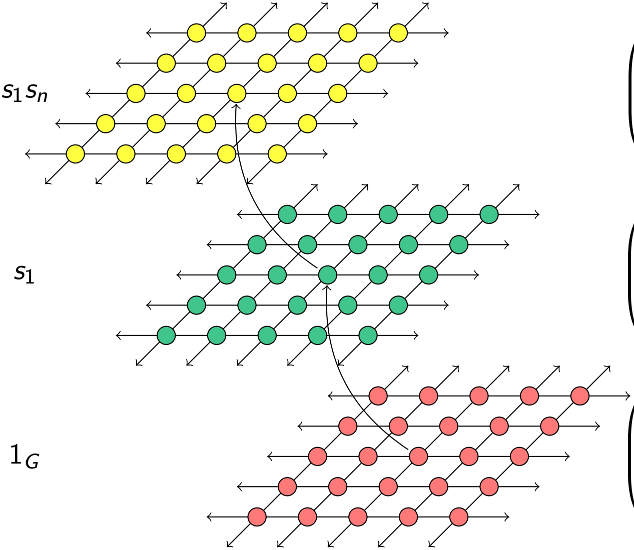


$$\begin{pmatrix} \Psi(z) \\ \Psi(T^{s_1}(z)) \\ \vdots \\ \Psi(T^{s_n}(z)) \end{pmatrix} \in \Pi$$

$$\begin{pmatrix} \Psi(y) \\ \Psi(T^{s_1}(y)) \\ \vdots \\ \Psi(T^{s_n}(y)) \end{pmatrix} \in \Pi$$

$$\begin{pmatrix} \Psi(x) \\ \Psi(T^{s_1}(x)) \\ \vdots \\ \Psi(T^{s_n}(x)) \end{pmatrix} \in \Pi$$

How does one prove such a thing?

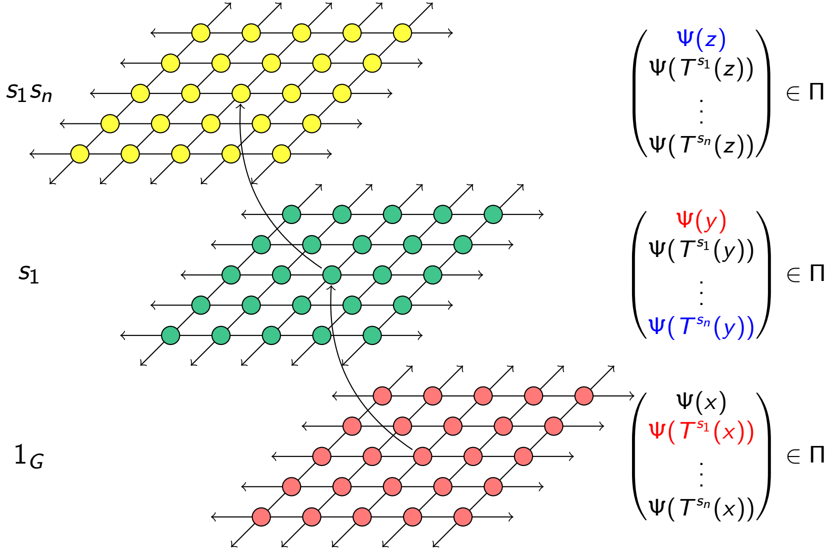


$$\begin{pmatrix} \Psi(z) \\ \Psi(T^{s_1}(z)) \\ \vdots \\ \Psi(T^{s_n}(z)) \end{pmatrix} \in \Pi$$

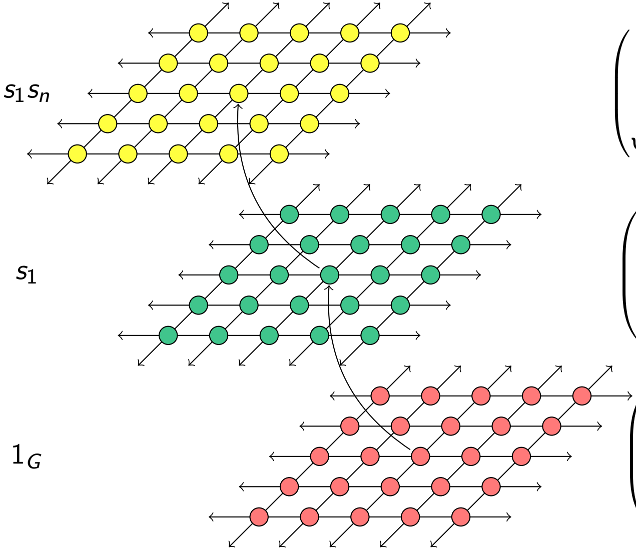
$$\begin{pmatrix} \color{red}{\Psi(y)} \\ \Psi(T^{s_1}(y)) \\ \vdots \\ \Psi(T^{s_n}(y)) \end{pmatrix} \in \Pi$$

$$\begin{pmatrix} \Psi(x) \\ \color{red}{\Psi(T^{s_1}(x))} \\ \vdots \\ \Psi(T^{s_n}(x)) \end{pmatrix} \in \Pi$$

How does one prove such a thing?



How does one prove such a thing?



$$\begin{pmatrix} \Psi(T^{s_1 s_n}(x)) \\ \Psi(T^{s_1 s_1 s_n}(x) \\ \vdots \\ \Psi(T^{s_n s_1 s_n}(x)) \end{pmatrix} \in \Pi$$

$$\begin{pmatrix} \Psi(T^{s_1}(x)) \\ \Psi(T^{s_1 s_1}(x)) \\ \vdots \\ \Psi(T^{s_1 s_n}(x)) \end{pmatrix} \in \Pi$$

$$\begin{pmatrix} \Psi(x) \\ \Psi(T^{s_1}(x)) \\ \vdots \\ \Psi(T^{s_n}(x)) \end{pmatrix} \in \Pi$$

From $\mathbb{Z}^2$ to $H_1 \times H_2$

How to go from $\mathbb{Z}^2$ to $H_1 \times H_2$?

From $\mathbb{Z}^2$ to $H_1 \times H_2$

How to go from $\mathbb{Z}^2$ to $H_1 \times H_2$?

[Whyte] translation-like action

an action $G \curvearrowright (X, d)$ is *translation-like* if:

- G acts freely
- For each $g \in G$, $\sup_{x \in X}(d(x, gx)) < \infty$.

From $\mathbb{Z}^2$ to $H_1 \times H_2$

How to go from $\mathbb{Z}^2$ to $H_1 \times H_2$?

[Whyte] translation-like action

an action $G \curvearrowright (X, d)$ is *translation-like* if:

- G acts freely
- For each $g \in G$, $\sup_{x \in X}(d(x, gx)) < \infty$.

Theorem (Seward, 2013)

Each infinite and f.g. group admits a translation-like action of $\mathbb{Z}$.

From $\mathbb{Z}^2$ to $H_1 \times H_2$

How to go from $\mathbb{Z}^2$ to $H_1 \times H_2$?

[Whyte] translation-like action

an action $G \curvearrowright (X, d)$ is *translation-like* if:

- G acts freely
- For each $g \in G$, $\sup_{x \in X}(d(x, gx)) < \infty$.

Theorem (Seward, 2013)

Each infinite and f.g. group admits a translation-like action of $\mathbb{Z}$.

This means that each infinite and f.g. group admits a Cayley graph that can be partitioned into disjoint bi-infinite paths.

From $\mathbb{Z}^2$ to $H_1 \times H_2$

Use the set of generators of the Cayley graph to define an SFT which codes the translation-like action.

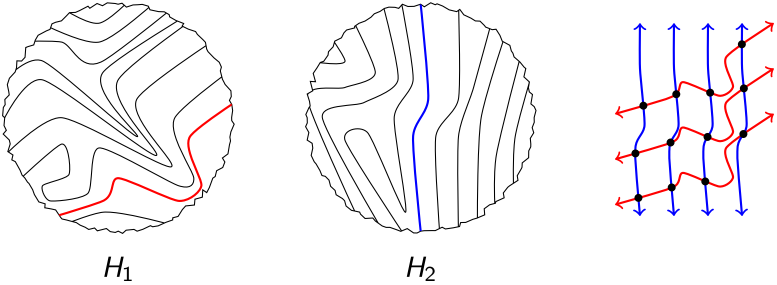


Figure: Finding a grid in $H_1 \times H_2$

Fin

Every EC $G \curvearrowright \{0,1\}^{\mathbb{N}}$ is a factor of a subaction of a $G \times H_1 \times H_2$-SFT

Every EC $\mathbb{Z}$-subshift is a subaction of a $\mathbb{Z}^2$-sofic

Every EC G-subshift is a subaction of a $G \times H_1 \times H_2$-sofic

$\exists$ SA EC subshifts in G

$\exists$ SA SFTs in $G_1 \times G_2 \times G_3$

$\exists$ SA SFTs in the Grigorchuk group

Fin

Theorem (B, 2017)

The Grigorchuk group admits a strongly aperiodic SFT.

Thank you for your attention!