Effectiveness in finitely generated groups.

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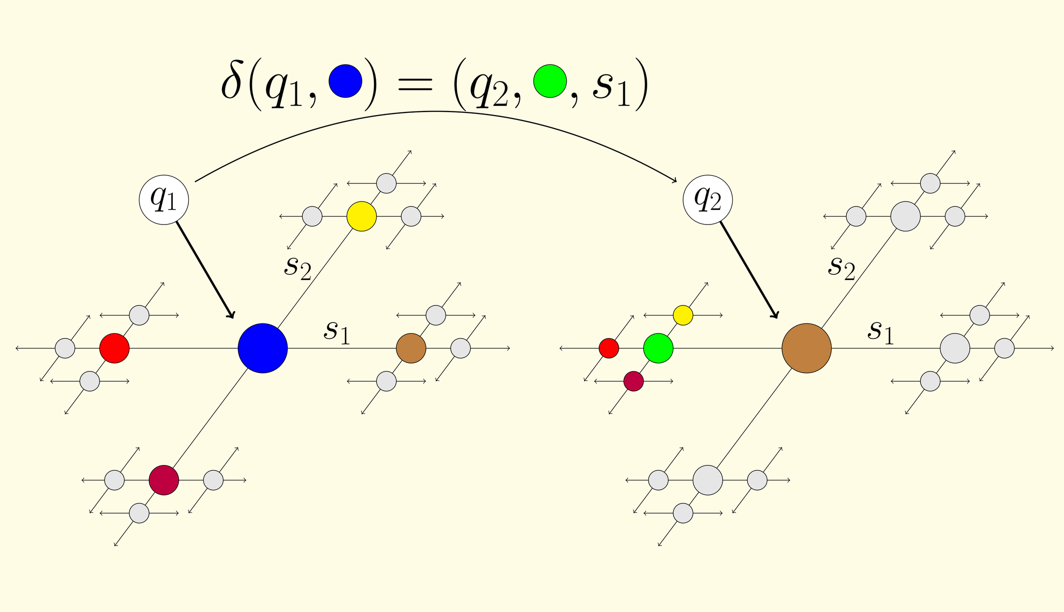
Background

Let G be a finitely generated group. A G-machine is a Turing machine whose tape has been replaced by the group G. The transition function is $\delta : Q \times \Sigma \to Q \times \Sigma \times (S \cup S^{-1} \cup \{1_G\})$ where S is a finite set of generators of G.

A transition in an F_2-machine.



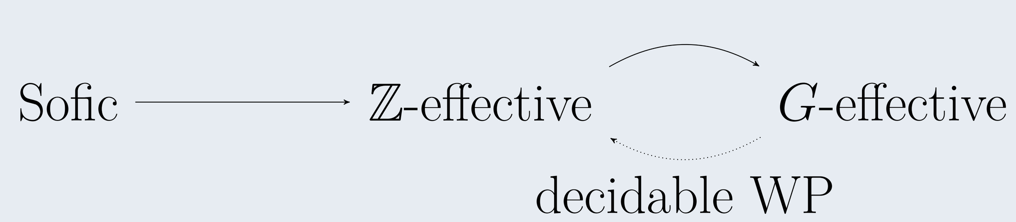
Let $\mathcal{A}$ be a finite alphabet. A set of patterns $\mathcal{F} \subset \mathcal{A}^*_G := \cup_{F \subset G, |F| < \infty} \mathcal{A}^F$ is said to be *G-recognizable* if there exists a G machine which reaches an accepting state for a pattern P if and only if $P \in \mathcal{F}$.

Definition: G-effectiveness

A G-subshift $X \subset \mathcal{A}^G$ is said to be *G-effective* if there exists a G-recognizable set $\mathcal{F} \subset \mathcal{A}^*_G$ such that $X = X_{\mathcal{F}} := \left\{ x \in \mathcal{A}^G \mid \forall P \in \mathcal{F}, P \not\sqsubset x \right\}$

Facts about G-effectiveness

- The class of G-effective subshifts is closed under factor codes.
- Every G-SFT and sofic G-subshift is G-effective.



Abstract

A general notion of effectiveness for finitely generated groups is presented by using a Turing machine which has a finitely generated group as a tape, giving thus a natural formulation of Turing machines which have access to an oracle of the word problem of that group. The class of G-effective subshifts is invariant under factors and is strictly larger than the one of G-sofic subshifts for some classes of groups.

What about the classical approach?

A *pattern coding* is a set of tuples $c = (w_i, a_i)_{1 \leq i \leq n}$ such that w_i is a word over a set of generators S of G and $a_i \in \mathcal{A}$. It is said to be consistent if words which represent the same element have the same symbol associated. A subshift $X \subset \mathcal{A}^G$ is said to be *$\mathbb{Z}$-effective* if there exists $\mathcal{F} \subseteq \mathcal{A}^*_G$ such that $X = X_{\mathcal{F}}$ and a Turing machine which enumerates a set of pattern codings such that the consistent ones codify $\mathcal{F}$.

Properties of $\mathbb{Z}$-effectiveness

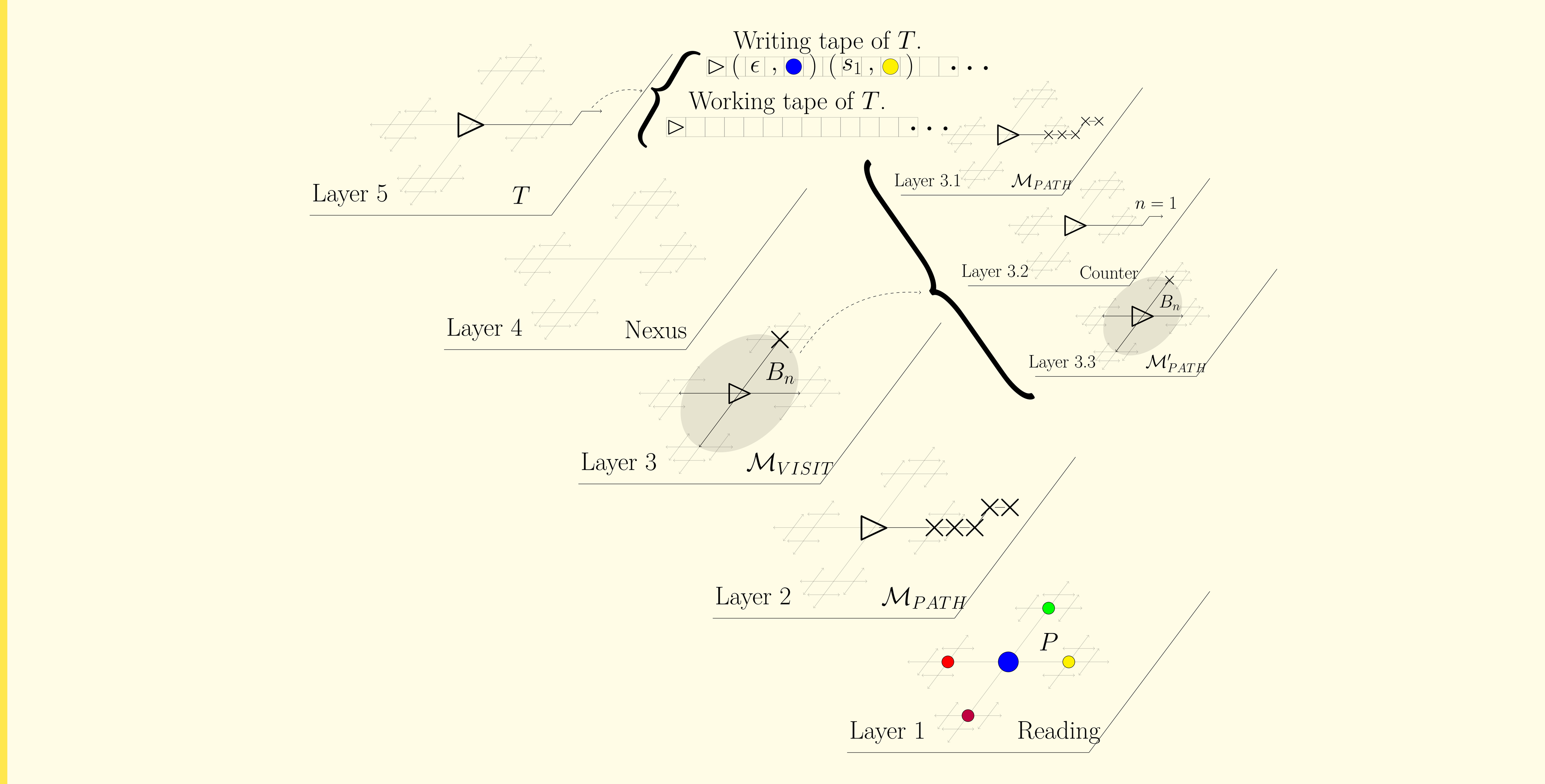
- If G is infinite and finitely generated every $\mathbb{Z}$-effective subshift is G-effective.
- If G has decidable word problem every G-effective subshift is $\mathbb{Z}$-effective.
- If G has undecidable word problem, there are G-effective subshifts which are not $\mathbb{Z}$-effective. One such example is the *one-or-less subshift*:

$$X_{\leq 1} = \left\{ x \in \{0,1\}^G \mid |\{g \in G : x_g = 1\}| \leq 1 \right\}.$$

Theorem

The class of G-effective subshifts is equal to the class of G-subshifts for which there is a Turing machine with oracle the word problem of G which satisfies the conditions of $\mathbb{Z}$-effectiveness.

Construction showing that $\mathbb{Z}$-effective subshifts are G-effective.



Reference

[1] N. Aubrun, S. Barbieri and M. Sablik. *A notion of effectiveness for subshifts on finitely generated groups*, arXiv:1412.2582.

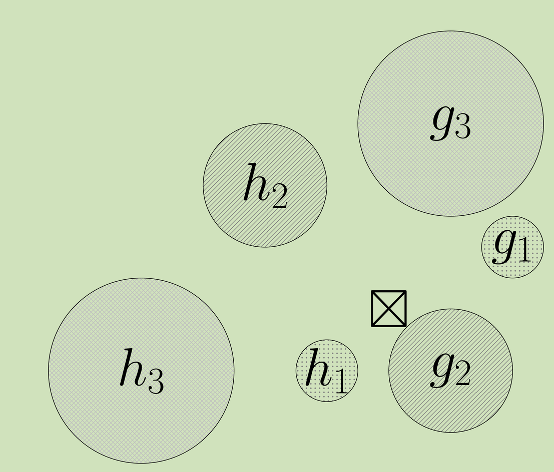
For which G are there G-effective but non-sofic G-subshifts?

First case: G recursively presented with undecidable word problem.

$X_{\leq 1}$ is G-effective but not sofic in this case.

Second case: G is amenable.

Using that $\inf_{F \subset G, |F| < \infty} |\partial F|/|F| = 0$ it is possible to adapt the proof that the mirror shift is non-sofic for the *ball mimic subshift*.



The ball mimic subshift is given by two sequences $(g_i)_{i \in \mathbb{N}}$ and $(h_i)_{i \in \mathbb{N}}$. If $\boxtimes$ appears in position $\bar{g}$, then $\forall i \in \mathbb{N}$, $x_{\bar{g}g_iB_i}$ and $x_{\bar{g}h_iB_i}$ coincide.

Third case: G has two or more ends.

If S generates G it is possible to disconnect $\Gamma(G, S)$ in two or more infinite connected components. One can force non-local constrains between the components to produce a counterexample.

Examples of groups where the answer is not known

Solutions to the Von Neumann conjecture provide non-amenable groups which do not contain a free group on two generators, thus one ended. The Tarski monster groups provide a continuum of non-isomorphic cases which don't fall in the categories above.