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Research Statement

Groups, Dynamics and Computability

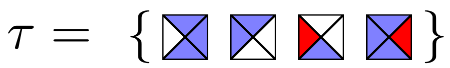
Keywords: Groups, dynamical systems, computability, symbolic dynamics, domino problem, entropy theory.

I am interested in three mathematical subjects which are quite often related:

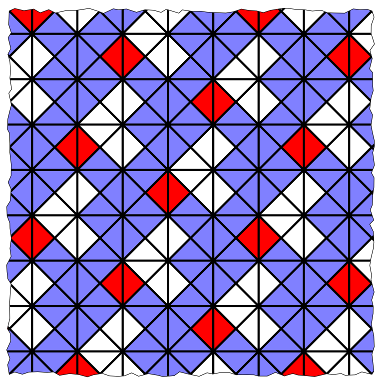
1. Countable groups and their geometrical and combinatorial properties.
2. Dynamical systems and ergodic theory, with a focus on symbolic dynamics.
3. Computability theory in the context of groups and dynamical systems.

One of the main angles of my research is to study the rich interplay between the dynamics of group actions and the computability properties of both the acting group and the action itself.

Let us illustrate this interplay with a historical example. In the sixties, Hao Wang was exploring satisfiability of a $\forall\exists\forall$ fragment of logic [Wan61]. From that, he derived the following problem: given a finite set τ of unit squares with colored edges and infinitely many copies of each square in that set, is there an algorithm which decides whether it is possible to tile the euclidean plane with those squares in such a way that every adjacent pair of tiles has the same color along their adjacent edge?



In the right we show a partial tiling of the euclidean plane using the four squares drawn above.



Wang did not answer his question, but gave the following interesting connection. We can consider the set of all tilings using the tileset τ as the closed subset X of all functions $x : \mathbb{Z}^2 \to \tau$ which respect the above constrains, and endow it with the action $\mathbb{Z}^2 \curvearrowright X$ which shifts every coordinate by a fixed element.

$$vx(u) := x(v + u) \text{ for every } u, v \in \mathbb{Z}^2 \text{ and } x \in X.$$

Wang showed that if every such action admits a periodic point, then the above problem would be decidable. It turned out that the problem was, in fact, undecidable [Ber66], and that there are tilesets which only admit aperiodic tilings.

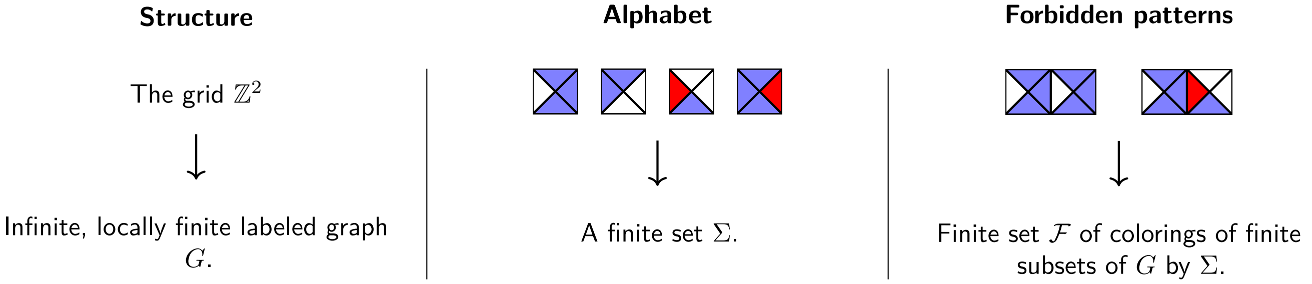
The previous example shows how a situation arising from computability bears impact on the dynamics: the undecidability of Wang's problem implies the existence of a specific type of free $\mathbb{Z}^2$-action. Far from being an isolated example, it seems to be a common situation: several dynamical properties have can be characterized by a computable property and vice-versa.

In what follows, I shall describe a few research directions I have been pursuing during my career.

Research directions

The domino problem on groups and graphs.

The question posed by Wang is now known under the name **domino problem**. A natural generalization of the domino problem is to modify the underlying euclidean plane structure for a more general metric space. A natural class of such spaces are infinite and locally finite labeled graphs. In this context, instead of considering colored square tiles we use an abstract finite set and replace the pasting rules by a finite set of forbidden colored finite connected subgraphs.



A generalized version of Wang's question for a fixed structure, is whether there exists an algorithm which given on input a finite alphabet and finite set of forbidden patterns decides whether there is a coloring of the structure which avoids all forbidden patterns. In the case where the structure is the Cayley graph of a finitely generated group, the decidability of the domino problem happens to be independent of the choice of Cayley graph, and thus an invariant of the group (See [ABJ18] for a proof). We can therefore speak unambiguously about the domino problem of a finitely generated group. Moreover, in the case of a group G the problem is shown to be equivalent to deciding whether a description of a subshift of finite type (SFT) over G is empty.

A reason to study this problem in finitely generated groups is the following. The Monadic second order (MSO) logic over the Cayley graph of a finitely generated group can be used to express its domino problem. Furthermore, it is known that MSO is decidable if and only if the group is context-free [KL05], and a group is virtually-free if and only if it is context-free and accessible [MS85]. Stated differently, (using that all finitely presented groups are accessible [Dun85]) groups that are not virtually free have an undecidable MSO logic. In particular, their result shows that virtually free groups have decidable domino problem. If the converse were true, it would mean that the domino problem would concentrate all the complexity of MSO logic for finitely generated groups.

Problem 1: Domino problem conjecture

Is it true that every finitely generated group has decidable domino problem if and only if it is virtually free.

The long term goal of this research direction is to settle this conjecture and study its generalization to graphs. So far it is known that the domino problem conjecture holds for the class of polycyclic groups [Jea15b], but there are solvable groups, such as the lamplighter group $\mathbb{Z} \wr \mathbb{Z}/2\mathbb{Z}$, for which the status of the conjecture is still open.

A strategy I have employed for studying this conjecture is to study this problem from a geometric perspective instead of a purely algebraic one. In a recent article [ABM18] we showed that for a class of graphs which can be parameterized by combinatorial structures, the domino problem is undecidable. We then apply this general result to show that the domino problem of the fundamental group of any closed surface of positive genus has undecidable domino problem, thus verifying the conjecture for a class of groups which is very "close" to virtually free groups. I have also used this angle with Sablik [BS16] to study the domino problem for self-similar graphs which arise from finite subdivision rules and show that there is no strict threshold in terms of the Hausdorff dimension for its undecidability.

Simulation theorems: free actions and entropy.

A simulation theorem is a result which ensures that every "complicated" computable object from a class can be embedded into a "simple" computable object from another class, often with an explicit finite presentation. An important example is the result by Higman [Hig61] which states that every recursively presented group can be obtained as a subgroup of a finitely presented group. An analogue in the category of dynamical systems is the simulation theorem of Hochman [Hoc09] which states that any effective action of $\mathbb{Z}$ over a closed and invariant subset of $\{0,1\}^{\mathbb{N}}$ can be realized as a factor of the subaction of a $\mathbb{Z}^3$-subshift of finite type.

A simulation result can shed light on complicated aspects of "simple" objects. In the setting of symbolic dynamics, the term "swamp of undecidability" [Lin04] was used to describe the lack of simple procedures to gain information about the properties of multidimensional subshifts of finite type. It was only after the result by Hochman and further developments [AS13, DRS10] that the origin of this swamp of undecidability was truly understood.

Another reason why simulation results are interesting is that they can be used as black boxes to easily show complicated results. For instance, the famous result by Novikov and Boone [Nov55, Boo58] that there exist finitely presented groups with undecidable word problem can be obtained as a relatively simple corollary from Higman's result. In the dynamical setting applications of simulation results include: the existence of aperiodic tilings in $\mathbb{Z}^d$ [Ber66, Rob71], the existence of $\mathbb{Z}^d$-subshifts of finite type without any computable configuration [Han74, Mye74] and a characterization of the topological entropies of $\mathbb{Z}^d$-subshifts of finite type [HM10].

My goal on this research direction is to search for generalizations of simulation theorems in the setting of group actions and exploit them to obtain information about their dynamics. Particularly, a long term goal is to find the full extent to which Hochman's theorem can be extended in the setting of finitely countable groups and explore its consequences over the existence of aperiodic subshifts of finite type and the entropies they can achieve.

Problem 2: Simulation theorems

Find suitable extensions of dynamical simulation theorems and use them to provide examples of systems exhibiting interesting dynamics.

In a recent contribution [BS18] we extended the simulation theorem of Hochman to show that any effective action of a finitely generated group G can be simulated as above in every semidirect product of the form $G \rtimes \mathbb{Z}^d$ for any $d \geq 2$. A consequence of this result is a complete classification of which polycyclic groups admit aperiodic subshifts of finite type [Jea15a]. I also showed a version [Bar17] of the simulation theorem which employs a technique that does not rely on having a component without torsion. As an application I characterized the infinite branch groups which admit free subshifts of finite type. In particular, I showed that the Grigorchuk group does.

Topological entropy

The topological entropy $h_{\text{top}}(\mathbb{Z} \curvearrowright X)$ of an action $\mathbb{Z} \curvearrowright X$ on a compact metrizable space X by homeomorphisms is topological invariant introduced by Adler, Konheim and McAndrew [AKM65] which measures the asymptotic growth of the number of distinguishable orbits on the system and constitutes "the most important" invariant of $\mathbb{Z} \curvearrowright X$. This theory (and its measurable counterpart) was later extended to the context of actions of countable amenable groups by Ornstein and Weiss [OW80] and other authors.

Recently, variants of this theory for actions of non-amenable groups have been developed. Notoriously, for sofic group actions the theory of measurable entropy was introduced by Bowen in [Bow10] and its topological counterpart along with a variational principle by Kerr and Li [KL11]. This new theory has provided tools to extend results from classical entropy theory to more general group actions.

I am interested in several aspects of entropy. One of them is the characterization, given a class of group actions, of the possible values that the topological entropies can take. For the class of subshifts of finite type, a classical result by Lind [Lin84] characterizes the topological entropies attainable by $\mathbb{Z}$-SFTs as non-negative rational multiples of logarithms of Perron numbers. On the other hand, a result by Hochman and Meyerovitch [HM10] shows that the characterization for $\mathbb{Z}^d$-SFTs for $d \geq 2$ is a computational one, namely, the numbers attained as entropies of $\mathbb{Z}^d$-SFTs coincides with the set of non-negative upper semi-computable real numbers. The same question can be asked for an arbitrary countable amenable group.

Problem 3: Topologial entropies of SFTs

Let G be a countable amenable group. Caracterize the set of real numbers that can be achieved as topological entropies of G-subshifts of finite type.

In a recent preprint [Bar19] I extended the classification from [Lin84, HM10] to all polycyclic-by-finite groups. I also showed that the characterization of Hochman and Meyerovitch holds for several classes of groups, notably including a large class of amenable branch groups with decidable word problem.

Another interesting direction comes from the local counterpart of entropy theory. Classical local entropy theory was initiated by Blanchard when he introduced the concept of entropy pairs [Bla93]. Later on Kerr and Li characterized entropy pairs using independence and developed the theory on sofic and general groups [KL07, KL13, KL16]. One advantage of local entropy theory is that it provides simple conditions that ensure positive (sofic) topological entropy. Interestingly, it has also revealed interesting connection between homoclinic pairs of $G \curvearrowright X$ and its entropy.

Problem 4: Entropy and homoclinic points

For which expansive actions $G \curvearrowright X$ does positive (sofic) topological entropy imply the existence of non-trivial homoclinic pairs?

In a recent article [BGR] we showed that for expansive actions of countable amenable groups which satisfy the pseudo-orbit tracing property (POTP) the homoclinic pairs and the entropy pairs are very closely related. In particular, we provided a local proof that every expansive action of a countable amenable group satisfying the POTP which has positive entropy admits non-trivial homoclinic pairs. In ongoing work with Felipe García Ramos and Hanfeng Li we show that our results hold under conditions weaker than POTP, and that in particular hold for several clases of finitely presented expansive algebraic actions.

Thermodynamic formalism and Markovian properties

I am interested in two notions that come from thermodynamic formalism and their relation. Namely, Gibbs measures and equilibrium measures. The notion of Gibbs measure is local, in the sense that they are the measures whose conditional expectations follow a distribution in which every finite region maximizes a function (Shannon entropy minus energy conditioned on the exterior) whereas equilibrium measures are defined by a global property (they maximize the measure theoretical entropy minus the energy per site).

The classical theorems of Dobrushin [Dob68] and Lanford-Ruelle [LR69] (DLR) give conditions for $\mathbb{Z}^d$-subshifts under which the equilibrium measures coincide with the Gibbs measures. While equilibrium measures can be complicated objects to describe, Gibbs measures can be described through a cocycle on the homoclinic pairs and are thus much easier to do computations with.

In [BGAMT18] we extended the DLR results to the context of countable amenable groups and extended the conditions under which these theorems hold in several directions. One of them is that we no longer require the underlying shift space to be of finite type, but only to satisfy a Markovian property. I am interested in exploring how these results extend in the context of non-amenable groups.

Problem 5: Gibbs and equilibrium measures in non-amenable groups

Under which conditions can results in the spirit of the DLR theorems be established for actions of non-amenable groups?

Some results of this nature have already been established [Bow19], although much still remains to be explored.

Groups arising from shift spaces

The automorphism group $\texttt{Aut}(G \curvearrowright X)$ of a group action consists of all G-equivariant homeomorphisms of X. It is a dynamical invariant in the sense that conjugate actions must have isomorphic automorphism groups. This object has turned out to be quite difficult to handle, in particular, it is still open whether for the shift action $\texttt{Aut}(\mathbb{Z} \curvearrowright \{0,1\}^{\mathbb{Z}})$ is conjugate to $\texttt{Aut}(\mathbb{Z} \curvearrowright \{0,1,2\}^{\mathbb{Z}})$ even if it is known that both groups embed into each other [MBR88]. Recently, there has been a plethora of results on automorphism groups in the case where X is a shift space and "not too complicated". For instance, in the case where X has low word complexity there are the results [CK15, DDMP16] which bound up to isomorphism the class of groups that can be obtained. There has also been new obstructions over which groups can be realized as an automorphism group of a subshift [CFKP16].

Another interesting group arising from a dynamical system $G \curvearrowright X$ is the topological full group $[[G \curvearrowright X]]$. This group is the set of all self-homeomorphisms ϕ of X for which $\phi(x) = g \cdot x$ for some $g \in G$. These groups are specially important in the theory of orbit equivalence of minimal actions. See [GPS99]. They have also given the first examples infinite finitely generated groups which are amenable and simple, see [Mat06, JM12].

Problem 6: Groups arising from shift spaces and their properties

What groups can be realized as an automorphism group or a topological full group of a subshift? what properties do they satisfy?

With Kari and Salo [BKS16] we introduced an abstract group of reversible Turing machines and showed that some computability invariants of this group, such as the word and torsion problems can be embedded into the automorphism group of a subshift. We used this to show that for any alphabet $|A| \geq 2$, $\texttt{Aut}(\mathbb{Z} \curvearrowright A^{\mathbb{Z}})$ and $[[\mathbb{Z}^d \curvearrowright A^{\mathbb{Z}^d}]]$ have undecidable torsion problem for $d \geq 2$.

Turing machines over discrete structures

In the classical model, the head of a Turing machine moves over a bi-infinite tape. However, one can generalize this model by making the machines move over a fixed, infinite, locally finite labeled graph. This has several applications, for instance, a way to define a notion of effectiveness for subshifts on groups with undecidable word problem, see [ABS17]. There are also dynamical objects which are naturally better represented in terms of these generalized machines, such as some cellular automata on groups [ST15]. In particular, it seems that a possible way to extend the results on the domino problem on surface groups to larger classes might need to encode the immortality problem of Turing machines that walk in larger classes of groups. Part of my ongoing work in this subject is to find suitable encodings for machines which walk on arbitrary graphs.

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