A hierarchy of topological systems with completely positive entropy

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Abstract

We define a hierarchy of systems with topological completely positive entropy in the context of countable amenable continuous group actions on compact metric spaces. For each countable ordinal we construct a dynamical system on the corresponding level of the aforementioned hierarchy and provide subshifts of finite type for the first three levels. We give necessary and sufficient conditions for entropy pairs by means of the asymptotic relation on systems with the pseudo-orbit tracing property, and thus create a bridge between a result by Pavlov and a result by Meyerovitch. As a corollary, we answer negatively an open question by Pavlov regarding necessary conditions for completely positive entropy.

1 Introduction

In ergodic theory a K-system is a measurable dynamical system that satisfies Kolmogorov's zero-one law. These systems were characterized by Rokhlin and Sinai as those where every non-trivial factor has positive entropy or as those where every non-trivial partition has positive entropy [RS61]. Blanchard introduced two topological analogues of the K-systems, that is, topological dynamical systems with uniformly positive entropy (UPE) and topological completely positive entropy (CPE) [Bla92], which can be defined respectively as those for which every standard open cover has positive entropy and those for which every non-trivial factor has positive entropy. Even though for some families, such as $\mathbb{Z}$-subshifts of finite type and expansive algebraic $\mathbb{Z}^d$-actions, UPE and topological CPE coincide, in general, topological CPE does not imply UPE.

In order to understand the properties of these systems, Blanchard introduced the notion of entropy pairs in [Bla93]. This seminal work is the birth of what is now called local entropy theory (see [GY09] for a survey). Loosely speaking, a pair of points is an entropy pair if every standard open cover which separates them has positive topological entropy. Blanchard showed (for $\mathbb{Z}$-actions) that a system has positive topological entropy if and only if there exists an entropy pair; a system has UPE if and only if every non-trivial pair is an entropy pair; and that the system has topological CPE if and only if the smallest closed equivalence relation containing the entropy pairs is the whole set X^2. These results were generalized to actions of countable amenable groups by Kerr and Li [KL07] by means of characterizing entropy pairs with the notion of independence. This point of view paved the way to a combinatorial study of topological entropy; even in the case of sofic groups actions. See Chapter 12 of [KL16].

In this paper we will introduce a hierarchy of topological dynamical systems that lie between UPE and topological CPE (see Section 2). A noteworthy remark is that the set of entropy pairs plus the diagonal is closed but not necessarily an equivalence relation. The hierarchy is defined as follows: the first level of the hierarchy corresponds to the systems where the entropy pairs and the diagonal are the whole product space, that is, systems with UPE; the second level consists of the systems which are not on the first level and such that the smallest equivalence relation that contains the entropy pairs is the whole product space. The third level is constituted by the systems which are neither on the first nor second level and such that the topological closure of the smallest equivalence relation that contains the entropy pairs is the whole product space. Subsequent levels of the hierarchy correspond to those systems for which the smallest closed equivalence relation containing the entropy pairs is the whole

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product space, but that require a larger number of alternating transitive and topological closures to stabilize. We show that in general this hierarchy does not collapse at any countable ordinal.

Theorem 3.3. *For every countable ordinal α the CPE class α is non-empty.*

Furthermore we show that in the first 3 levels of the hierarchy we can find $\mathbb{Z}^d$-SFTs (Corollary 5.13 and Theorem 6.21). This exemplifies how rich, from the topological dynamics point of view, is the class of SFTs. In order to construct these examples we first needed to better understand entropy pairs in the setting of SFTs or more generally on dynamical systems with the pseudo-orbit tracing property. It turns out that they are closely related to the notion of asymptotic pairs.

We say (x, y) is an asymptotic pair if for every $\varepsilon > 0$ the set of elements g in the group for which $d(gx, gy) > \varepsilon$ is finite. In [CL14] Chung and Li asked if every expansive action of a countable amenable group with positive topological entropy has a non-trivial (i.e. $x \neq y$) asymptotic pair. Recently Meyerovitch [Mey] showed that in general this is not true. Nonetheless there are two families: an algebraic one [CL14][LS99] and expansive systems with the pseudo-orbit tracing property (POTP) [Mey] where the result holds.

In this paper, we characterize exactly how the local theory of entropy and asymptotic pairs are related in the context of countable amenable group actions with the pseudo-orbit tracing property. We give a localized version of Meyerovitch's result by using the formalism of entropy pairs and show that a sort of converse also holds.

Let us state our result precisely. A G-topological dynamical system (G-TDS) is a pair (X, T) where X is a compact metric space and T is a left G-action on X by homeomorphisms. Denote the diagonal of X^2 by Δ, the set of entropy pairs by $\mathtt{E}(X, T)$ and by $\mathtt{A}^\varepsilon(X, T)$ the set of pairs (x, y) for which $d(gx, gy) > \varepsilon$ for finitely many $g \in G$ and by $\mathtt{A}(X, T) = \bigcap_{\varepsilon>0} \mathtt{A}^\varepsilon(X, T)$ the set of asymptotic pairs. Let G be a countable amenable group. Our result can be stated as follows:

Theorem 4.2. *Let (X, T) be a G-TDS with the pseudo-orbit tracing property.*

1. *If $(x, y) \in \mathtt{E}(X, T)$ then $(x, y) \in \overline{\mathtt{A}^\varepsilon(X, T)} \setminus \Delta$ for every $\varepsilon > 0$ and there exists an invariant measure μ such that $x, y \in \operatorname{supp}(\mu)$.*

2. *If $(x, y) \in \mathtt{A}(X, T)$ and there exists an invariant measure μ such that $x, y \in \operatorname{supp}(\mu)$ then $(x, y) \in \mathtt{E}(X, T) \cup \Delta$.*

Even though the hierarchy is an abstract construction this theorem provides a very concrete way to look at entropy pairs. For instance, it tells us that UPE systems with the pseudo-orbit tracing property and a fully supported measure have a dense set of asymptotic pairs. This theorem is also the main tool used in this paper to check if certain particular examples have UPE or CPE and to determine what precise class they belong to.

In the case where (X, T) is expansive, we have that $\mathtt{A}(X, T) = \mathtt{A}^\varepsilon(X, T)$ for some positive ε. In particular, Theorem 4.2 allows us to recover Meyerovitch's result.

Other results that relate asymptotic pairs and positive topological entropy are those of Pavlov. In [Pav13] Pavlov characterized $\mathbb{Z}^d$-SFTs with a fully supported measure having only positive entropy symbolic factors through a combinatorial condition on the patterns, called chain exchangability (CE); loosely speaking, a subshift has chain exchangability if every pattern can be turned into any other pattern over the same support by constructing a chain of patterns such that any two consequent patterns on the chain can be obtained by localizing each of the configurations on an asymptotic pair. Subsequently, in [Pav18], he gave sufficient conditions which imply topological CPE for $\mathbb{Z}^d$-SFTs; his condition being bounded chain exchangeability (BCE). In the same paper, Pavlov asked whether the condition of having BCE was necessary to have topological CPE in the context of $\mathbb{Z}^d$-subshifts of finite type.

As another consequence of Theorem 4.2 we show that any subshift of finite type satisfying Pavlov's BCE condition (with a fully supported measure) must belong to the first or second level of our hierarchy of complete entropy. In particular, this gives a new proof of Pavlov's result and extends it to countable amenable groups. Furthermore, we answer his question in the negative by constructing an SFT that we denote as the Good Wave Shift.

Theorem 6.22. *There is a topologically weakly mixing $\mathbb{Z}^3$-SFT with topological CPE which does not have BCE.*

Remark 1.1. Two important developments have occurred since the initial version of this paper appeared. First, Salo [Sal19] proved a stronger version of Theorem 3.3. More precisely, he showed that the hierarchy does not collapse for $\mathbb{Z}$-subshifts (in our construction the systems are zero-dimensional but not expansive). Second, Westrick [Wes19] gave an astounding positive answer to Question 6.23 by constructing a $\mathbb{Z}^2$-SFT in the CPE class α for every computable ordinal α. Furthermore, she showed that if a family has members in every computable CPE class then the topological CPE property is coanalytic complete (on that family). This prescribes the possibility of any "simple" characterization of topological CPE in the family of SFTs such as BCE.

2 Background

Let G be a group. We denote by $F \Subset G$ a finite subset of G. A sequence $\{F_n\}_{n\in\mathbb{N}}$ of finite subsets of G is said to be (left) **asymptotically invariant** or **Følner** if for every $K \Subset G$ we have that $|K \cdot F_n \Delta F_n|/|F_n| \to 0$. A countable group is **amenable** if it admits a Følner sequence.

From now on, G denotes an arbitrary countable amenable group. A **G-topological dynamical system (G-TDS)** is a pair (X,T) where X is a compact metric space and $T : G \times X \to X$ is a left G-action on X by homeomorphisms $T(g,x) = T^g(x)$. We say that a system (Y,S) is a **factor** of (X,T) if there exists a continuous surjective G-equivariant map $\phi : X \to Y$. For an open cover $\mathcal{U}$ of X, and $F \Subset G$ we denote by $\mathcal{U}^F = \bigvee_{g\in F} T^{g^{-1}}\mathcal{U}$ the refinement of $\mathcal{U}$ by F. We also denote the minimum cardinality of a subcover of $\mathcal{U}$ by $N(\mathcal{U})$.

2.1 Entropy pairs and the CPE class

Definition 2.1. *Let (X,T) be a G-TDS, $\mathcal{U}$ an open cover and $\{F_n\}_{n\in\mathbb{N}}$ a Følner sequence. We define the **topological entropy of (X,T) with respect to $\mathcal{U}$** as*

$$h_{top}(T,\mathcal{U}) = \lim_{n\to\infty} \frac{1}{|F_n|} \log N(\mathcal{U}^{F_n}).$$

Note that this limit does not depend on the choice of Følner sequence, see for instance Theorem 4.38 in [KL16]. The **topological entropy** of (X,T) is defined as

$$h_{\mathrm{top}}(T) = \sup_{\mathcal{U}} h_{\mathrm{top}}(T,\mathcal{U}).$$

Definition 2.2. *We say a G-TDS (X,T) has **uniform positive entropy (UPE)** if for each standard (two non-dense open sets) cover $\mathcal{U}$ we have that $h_{top}(T,\mathcal{U}) > 0$. We say it has **topological completely positive entropy (topological CPE)** if each non-trivial factor has positive topological entropy.*

In general we have that every system with UPE has topological CPE but the converse is not true even if the system is minimal [SY09].

Definition 2.3. *Let (X,T) be a G-TDS. A pair $(x,y) \in X^2$ is an **entropy pair** if $x \neq y$ and for every pair of disjoint closed neighborhoods U_x, U_y of x and y respectively we have that*

$$h_{top}(T, \{U_x^{\mathsf{c}}, U_y^{\mathsf{c}}\}) > 0.$$

The set of entropy pairs is denoted by $\mathtt{E}(X,T)$.

Theorem 2.4. *[[Bla93][KL07]] Let (X,T) be a G-TDS and $\Delta = \{(x,x) \mid x \in X\}$ the diagonal of X^2.*

1. *There exists an entropy pair if and only if $h_{top}(T) > 0$.*
2. *$\mathtt{E}(X,T) \cup \Delta = X^2$ if and only if (X,T) has UPE.*
3. *The smallest closed equivalence relation that contains the entropy pairs is X^2 if and only if (X,T) has topological CPE.*

Given a subset $R \subset X^2$ we denote with R^+ the smallest equivalence relation that contains R. Considering that the union of the set of entropy pairs and the diagonal is closed but not an equivalence relation, we define the following sets inductively (transfinitely). Let α be an ordinal. We define

$$\mathtt{E}_1(X,T) := \mathtt{E}(X,T) \cup \Delta.$$

$$\text{If } \alpha \text{ is a successor} \quad \mathtt{E}_\alpha(X,T) := \begin{cases} \overline{\mathtt{E}_{\alpha-1}(X,T)} & \text{if } \mathtt{E}_{\alpha-1}(X,T) \text{ is not closed,} \\ \mathtt{E}_{\alpha-1}(X,T)^+ & \text{otherwise.} \end{cases}$$
$$\text{If } \alpha \text{ is a limit} \quad \mathtt{E}_\alpha(X,T) := \bigcup_{\beta<\alpha} \mathtt{E}_\beta(X,T).$$

The following definition introduces a hierarchy of systems that lie between UPE and topological CPE.

Definition 2.5. *Let α be an ordinal. We say that a dynamical system (X,T) is in the* ***CPE class*** *α if $E_\alpha(X,T) = X^2$ and for every $\beta < \alpha$ we have $E_\beta(X,T) \neq X^2$.*

Note that (X,T) is in the CPE class 1 if and only if (X,T) has UPE.

Proposition 2.6. *A G-TDS (X,T) is in the CPE class α for some countable ordinal α if and only if it has topological CPE.*

Proof. If (X,T) is in the CPE class α then the smallest closed equivalence relation containing $E(X,T)$ is X^2 therefore by Theorem 2.4 it has topological CPE. Conversely, by the same result, the smallest closed equivalence relation containing $E(X,T)$ is X^2. As the increasing chain $\{\overline{E_\alpha(X,T)}\}_\alpha$ of closed sets is contained in X^2 which is separable, it must stabilize at a countable ordinal. See for instance Chapter 1, exercise 18 of [Aki10]. □

As far as the authors are aware, every example in the literature of a G-TDS which has topological CPE but not UPE is either in the CPE class 2 or 3. For instance, Blanchard's example from [Bla92] $X = \{a,b\}^{\mathbb{Z}} \cup \{a,c\}^{\mathbb{Z}}$ with the shift action clearly is in the CPE class 2. And Song and Ye's minimal example from [SY09] is diagonal, that is, $(x,Tx) \in E(X,T)$ for every $x \in X$. Therefore it satisfies that $\overline{E(X,T)^+} = X^2$, meaning that it cannot belong to any class above 3. We will later prove that for every countable ordinal the corresponding CPE class is non-empty. Furthermore, we will also show the existence of subshifts of finite type in the first three classes.

2.2 Asymptotic pairs and the asymptotic class

Here we define a similar class but based on asymptotic pairs. The main motivation for introducing this class is that under some conditions given in Section 4, the asymptotic class and the CPE class coincide. Nonetheless, since asymptotic pairs on dynamical systems are natural objects which can be used to study chaotic behaviour (see for references [BHR02, DL11, HXY15]), we believe the asymptotic class is interesting on its own.

Definition 2.7. *Let (X,T) a G-TDS. We say (x,y) is an* ***ε-asymptotic pair*** *if there exists $F \Subset G$ such that for $g \notin F$ $d(T^g x, T^g y) \leq \varepsilon$. Furthermore, we say (x,y) is an* ***asymptotic pair*** *if it is ε-asymptotic for every $\varepsilon > 0$.*

We denote the ε-asymptotic pairs with $\mathtt{A}^\varepsilon(X,T)$ and the asymptotic pairs with $\mathtt{A}(X,T)$. The asymptotic pairs form an equivalence relation that in general is not closed. Let (X,T) be a G-TDS. For an ordinal α we define the following increasing set of asymptotic relations.

$$\mathtt{A}_0(X,T) := \mathtt{A}(X,T), \text{ and}$$

$$\text{If } \alpha \text{ is a successor} \quad \mathtt{A}_\alpha(X,T) := \begin{cases} \overline{\mathtt{A}_{\alpha-1}(X,T)} & \text{if } \mathtt{A}_{\alpha-1}(X,T) \text{ is not closed,} \\ \mathtt{A}_{\alpha-1}(X,T)^+ & \text{otherwise.} \end{cases}$$
$$\text{If } \alpha \text{ is a limit} \quad \mathtt{A}_\alpha(X,T) := \bigcup_{\beta<\alpha} \mathtt{A}_\beta(X,T)$$

For an ordinal α we say that a G-TDS (X,T) **is in the asymptotic class** α if $\mathtt{A}_\alpha(X,T) = X^2$ and for every $\beta < \alpha$ we have $\mathtt{A}_\beta(X,T) \neq X^2$. We remark that every system for which the smallest closed equivalence relation containing $\mathtt{A}(X,T)$ is X^2 must satisfy that $\mathtt{A}_\alpha(X,T) = X^2$ for some countable ordinal α by analogous reasons as those given in Proposition 2.6.

2.3 Independence

The notion of independence can be used to characterize entropy pairs. A suggested reference for this topic is Chapter 12 of Kerr and Li's book [KL16].

Definition 2.8. *We say $J \subset G$ has* ***positive density*** *$D(J)$* ***with respect to a Følner sequence*** *$\{F_n\}_{n\in\mathbb{N}}$ if the following limit exists and is positive*

$$D(J) := \lim_{n\to\infty} \frac{|F_n \cap J|}{|F_n|} > 0.$$

Definition 2.9. *Let (X,T) be a G-TDS. Let $\mathbf{A} = (A_1, \ldots, A_n)$ be a tuple of subsets of X. We say $J \subset G$ is an* ***independence set*** *for $\mathbf{A}$ if for every nonempty $I \Subset J$ and any $\phi : I \to \{1, \ldots, n\}$ we have*

$$\bigcap_{i\in I} T^{i^{-1}} A_{\phi(i)} \neq \emptyset$$

Definition 2.10. *We say (x,y) is an* ***independence entropy pair (IE-pair)*** *if every pair of open sets $\mathbf{U} = \{U_1, U_2\}$ with $x_1 \in U_1$ and $x_2 \in U_2$, admits an independence set J with positive density for some Følner sequence. We denote the set of independence entropy pairs with $\mathtt{IE}(X,T)$.*

Definition 2.11. *Let (X,T) be a G-TDS and μ an invariant measure. We say a Følner sequence $\{F_n\}_{n\in\mathbb{N}}$* ***satisfies the pointwise ergodic theorem (PET)*** *if for every $f \in L^1(X)$ we have that*

$$\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{g\in F_n} f \circ T^g x = \mathbb{E}(f)(x). \quad \mu\text{-a.e.}$$

Where $\mathbb{E}$ is the conditional expectation of μ with respect to the subspace of G-invariant functions in $L^1(X)$.

Remark 2.12. Every countable amenable group admits a Følner sequence which satisfies the PET. This was first proven by Lindenstrauss [Lin01] using the notion of tempered Følner sequence of [Shu88].

Remark 2.13 (Proposition 12.7 of [KL16])**.** Let (x,y) be an IE-pair. For every Følner sequence $\{F_n\}_{n\in\mathbb{N}}$ that satisfies the PET there exists an independence set with positive density with respect to $\{F_n\}_{n\in\mathbb{N}}$.

It can be shown that the set of $x \in X$ such that $(x,x) \in \mathtt{IE}(X,T)$ coincides with the set of all $x \in X$ for which there exists an invariant measure μ for which $x \in \mathrm{supp}(\mu)$, see Proposition 3.12 of [KL07]. Let us denote the above set by $\mathcal{S}(X,T)$ and let $\Delta_{\mathcal{S}(X,T)} = \{(x,x) \in X^2 : x \in \mathcal{S}(X,T)\}$ be the diagonal of $\mathcal{S}(X,T)$.

Theorem 2.14. *[[KL07] or Theorem 12.20 [KL16]] Let (X,T) be a G-TDS. Then $\mathtt{IE}(X,T) = \mathtt{E}(X,T) \cup \Delta_{\mathcal{S}(X,T)}$.*

2.4 Pseudo-orbit tracing property

Definition 2.15. *Let (X,T) a G-TDS, $\delta > 0$ and $S \Subset G$. An (S,δ)* ***pseudo-orbit*** *is a sequence $(x_g)_{g\in G}$ such that $d(T^s x_g, x_{s\cdot g}) < \delta$ for all $s \in S$ and $g \in G$. We say a pseudo-orbit is ε-**traced by** x if $d(T^g x, x_g) \leq \varepsilon$ for all $g \in G$.*

Definition 2.16. *A G-TDS (X,T) has the* ***pseudo-orbit tracing property (POTP)*** *if for every $\varepsilon > 0$ there exists $\delta > 0$ and a finite set S such that any (S,δ) pseudo-orbit is ε-traced by some point $x \in X$.*

This condition is also known as shadowing. In the case where the space X is zero-dimensional, any expansive G-TDS (X,T) is topologically conjugate to a subshift. In this context, the systems satisfying the POTP are exactly the subshifts of finite type (SFTs). That is, they can be seen as sets of coloring of the group G by a finite alphabet which can be characterized by forbidding the appearance of a finite set of patterns. This equivalence was first shown by Walters [Wal78] in the context of $\mathbb{Z}$-actions and subsequently generalized to $\mathbb{Z}^d$ by Oprocha [Opr08] and arbitrary finitely generated groups by Chung and Lee [CL17]. This last result also covers the arbitrary countable case, as remarked by Meyerovitch [Mey].

Other families of systems satisfying the POTP are axiom A diffeomorphisms [Bow78] and principal algebraic actions of countable amenable groups [Mey].

3 Examples for every CPE class

Lemma 3.1. *Let α be an ordinal and (X,T),(Y,S) two G-TDS. If $\phi : X \longrightarrow Y$ is a G-equivariant continuous map then $\phi(\mathtt{E}_\alpha(X,T)) \subset \mathtt{E}_\alpha(Y,S)$.*

Proof. We proceed using transfinite induction. The base step is already known, see [Bla93]. However, we will do the proof for completeness. Let $(x,y) \in \mathtt{E}_1(X,T)$. If $\phi(x) = \phi(y)$ then clearly $(\phi(x),\phi(y)) \in \mathtt{E}_1(Y,S)$. Assume $\phi(x) \neq \phi(y)$ and let $V_{\phi(x)}, V_{\phi(y)}$ be disjoint closed neighborhoods of $\phi(x)$ and $\phi(y)$ respectively. Since $(x,y) \in \mathtt{E}_1(X,T)$ we have that

$$h_{top}\left(T, \left\{\left(\phi^{-1}V_{\phi(x)}\right)^c, \left(\phi^{-1}V_{\phi(x)}\right)^c\right\}\right) > 0.$$

By continuity and G-equivariance of ϕ we have that

$$h_{top}\left(S, \left\{\left(V_{\phi(x)}\right)^c, \left(V_{\phi(x)}\right)^c\right\}\right) = h_{top}\left(T, \left\{\left(\phi^{-1}V_{\phi(x)}\right)^c, \left(\phi^{-1}V_{\phi(x)}\right)^c\right\}\right).$$

Therefore $(\phi(x),\phi(y)) \in \mathtt{E}_1(Y,S)$. Now let α be an ordinal and assume that $\phi(\mathtt{E}_\beta(X,T)) \subset \mathtt{E}_\beta(Y,S)$ for every $\beta < \alpha$.

It is not hard to see that if $\phi(A) \subset B$ then $\phi(\overline{A}) \subset \overline{B}$ and $\phi(A^+) \subset B^+$. This proves the result for any successor ordinal α. In the case of a limit ordinal we have that $\phi(A_\beta) \subset B_\beta$ and therefore $\phi(A_\alpha) = \phi(\cup_{\beta<\alpha} A_\beta) \subset \cup_{\beta<\alpha} B_\beta = B_\alpha$. Hence the result also holds for α limit. □

Remark 3.2. If we consider the asymptotic class in Lemma 3.1 instead of the CPE class the result still holds. The image under a G-equivariant continuous map of an asymptotic pair is clearly asymptotic, and the rest of the argument is the same.

We will define a family of dynamical systems (X_α, T_α) for every countable ordinal α and a pair of fixed points x_α, y_α whenever α is a successor. Let $X_1 := \{0,1\}^G$, $T_1 := \sigma : G \times X \longrightarrow X$ denote the left shift G-action on X where for every $g \in G$ we have $\sigma^g(x)_h = x_{g^{-1}h}$ for every $h \in G$, and $x_1 = 0^G$, $y_1 = 1^G$ be fixed points for σ. Let $I := \{\frac{1}{n} : n > 0\} \cup \{0\}$ with the Euclidean topology τ_I. We schematize the four cases of the construction in Figure 1

1. If α is an even successor ordinal we define

$$X_\alpha := X_{\alpha-1} \times \{1,2,3\} \diagup \left[(y_{\alpha-1},1) \sim (y_{\alpha-1},2) \wedge (x_{\alpha-1},2) \sim (x_{\alpha-1},3)\right],$$

$$x_\alpha := (x_{\alpha-1},1),\ y_\alpha := (y_{\alpha-1},3)$$

with $\tau_\alpha = \tau_{\alpha-1} \times 2^{\{1,2,3\}}$ the product topology and $T^g_\alpha : X_\alpha \longrightarrow X_\alpha$ the map

$$T^g_\alpha((x,i)) := (T^g_{\alpha-1}(x), i) \text{ for every } x \in X_{\alpha-1} \text{ and } i \in \{1,2,3\}.$$

2. If α is an odd successor ordinal and $\alpha - 1$ is not a limit ordinal we define

$$X_\alpha := X_{\alpha-1} \times \{I\} \diagup \begin{bmatrix} (y_{\alpha-1}, \frac{1}{n}) \sim (y_{\alpha-1}, \frac{1}{n+1}),\ \forall n \text{ odd} \\ (x_{\alpha-1}, \frac{1}{n}) \sim (x_{\alpha-1}, \frac{1}{n+1}),\ \forall n > 0 \text{ even} \end{bmatrix},$$

$$x_\alpha := (x_{\alpha-1},1),\ y_\alpha := (y_{\alpha-1},0),$$

with $\tau_\alpha = \tau_{\alpha-1} \times \tau_I$ the product topology and $T^g_\alpha : X_\alpha \longrightarrow X_\alpha$ the map

$$T^g_\alpha((x,i)) := (T^g_{\alpha-1}(x), i) \text{ for every } x \in X_{\alpha-1} \text{ and } i \in I.$$

3. If α is a countable limit ordinal, fix an increasing sequence of successor ordinals $\{\alpha(n)\}_{n\in\mathbb{N}}$ such that $\alpha(n) \longrightarrow \alpha$ and define

$$X_\alpha := \bigsqcup_{n\in\mathbb{N}} X_{\alpha(n)} \diagup \left[y_{\alpha(n)} \sim y_{\alpha(n+1)},\ \forall n \in \mathbb{N}\right].$$

Here $\bigsqcup$ stands for disjoint union. We consider the topology $\tau_\alpha = \tau^1_\alpha \cup \tau^2_\alpha$ where

$\tau^1_\alpha :=\{U$: there is $n \in \mathbb{N}$ such that $U \in \tau_{\alpha(n)}$ and $y_{\alpha(n)} \notin U\}$

$\tau^2_\alpha :=\{V$: there are $n, m \in \mathbb{N}$ such that $V = (X_\alpha \setminus \bigsqcup_{k<n} X_{\alpha(k)}) \cup U$ and $y_{\alpha(m)} \in U \in \tau_{\alpha(m)}\}$.

The topology is explained in Figure 2. Here $T^g_\alpha(x) = T^g_{\alpha(n)}(x)$ if $x \in X_{\alpha(n)}$.

4. If $\alpha - 1$ is a limit ordinal, let $\{\alpha(n)\}_{n\in\mathbb{N}}$ be the sequence used to define $X_{\alpha-1}$. We define

$$X_\alpha := X_{\alpha-1} \times \{I\} \big/ \left[(x_{\alpha(n)}, \tfrac{1}{n}) \sim (x_{\alpha(n)}, \tfrac{1}{n+1}),\ \forall n > 0\right],$$

$$x_\alpha := (x_{\alpha(0)}, 1),\ y_\alpha := (x_{\alpha(0)}, 0)$$

with $\tau_\alpha = \tau_{\alpha-1} \times \tau_I$ the product topology and $T^g_\alpha : X_\alpha \longrightarrow X_\alpha$ the map

$$T^g_\alpha((x,i)) := (T^g_{\alpha-1}(x), i) \ \text{ for every } x \in X_{\alpha-1} \text{ and } i \in I.$$

One can check that on every step X_α is compact, Hausdorff and second countable with the topology τ_α, and hence metrizable. Also it is not hard to see that T_α is a continuous left G-action.

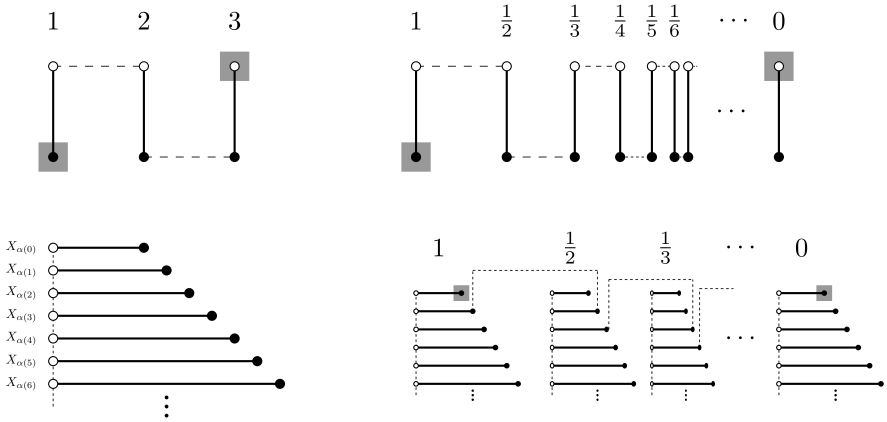


Figure 1: The construction in each of the four cases defined above. The black dots and white dots represent $x_{\alpha-1}$ and $y_{\alpha-1}$ respectively. The dashed lines represent the identified points and the boxes the new x_α and y_α.

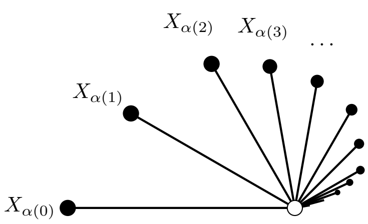


Figure 2: The topology τ_α in the case α is a limit ordinal is better understood as follows: since each $X_{\alpha(n)}$ is zero-dimensional it can be represented as a subset of the unit interval. τ_α is the relative topology of the embedding in $\mathbb{R}^2$ of all $X_{\alpha(n)}$ as intervals of length $\frac{1}{n+1}$ which intersect in $y_{\alpha(n)}$.

Theorem 3.3. *For every countable ordinal α the CPE class α is non-empty.*

Proof. By transfinite induction we will show that (X_α, T_α) is in the CPE class α. We will use the following induction hypothesis on all $\beta < \alpha$.

(a) $\mathtt{E}_\beta(X_\beta, T_\beta) = X^2_\beta$.

(b) If β is successor, we have that $(x_\beta, y_\beta) \notin \mathtt{E}_{\beta-1}(X_\beta, T_\beta)$.

(c) If β is a countable limit ordinal, for every $n > 0$, we have $(x_{\beta(0)}, x_{\beta(n+1)}) \notin \mathtt{E}_{\beta(n)}(X_\beta, T_\beta)$.

Clearly, showing these hypothesis is enough to prove the theorem. The base cases $\mathtt{E}_1(X_1, T_1)$ and $\mathtt{E}_2(X_2, T_2)$ are direct and thus there are only four cases to consider. We bring to the attention of the reader that in order to prove that (b) and (c) holds we shall proceed by contradiction.

Case 1: α is an even successor ordinal.

(a) Let ϕ_i be the natural embedding that sends $X_{\alpha-1}$ to X_α by $\phi_i(x) = (x, i)$. This is clearly a continuous G-equivariant map, therefore using Lemma 3.1 and $\mathtt{E}_{\alpha-1}(X_{\alpha-1}, T_{\alpha-1}) = X^2_{\alpha-1}$ we obtain that

$$(X_{\alpha-1} \times \{1\})^2 \cup (X_{\alpha-1} \times \{2\})^2 \cup (X_{\alpha-1} \times \{3\})^2 \subset \mathtt{E}_{\alpha-1}(X_\alpha, T_\alpha).$$

As $\mathtt{E}_\alpha(X_\alpha, T_\alpha)$ is transitive, $(y_{\alpha-1}, 1) \sim (y_{\alpha-1}, 2)$ and $(x_{\alpha-1}, 2) \sim (x_{\alpha-1}, 3)$ then $\mathtt{E}_\alpha(X_\alpha, T_\alpha) = X^2_\alpha$.

(b) Suppose $((x_{\alpha-1}, 1), (y_{\alpha-1}, 3)) \in \mathtt{E}_{\alpha-1}(X_\alpha, T_\alpha)$. As $\mathtt{E}_{\alpha-1}(X_\alpha, T_\alpha)$ is closed we can find sequences $\{(x^{(n)}, i^{(n)})\}_{n\in\mathbb{N}} \longrightarrow (x_{\alpha-1}, 1)$, $\{(y^{(n)}, j^{(n)})\}_{n\in\mathbb{N}} \longrightarrow (y_{\alpha-1}, 3)$ such that $((x^{(n)}, i^{(n)}), (y^{(n)}, j^{(n)})) \in \mathtt{E}_{\alpha-2}(X_\alpha, T_\alpha)$ for each $n \in \mathbb{N}$. Define the projection map $\Psi_{1,3} : X_\alpha \to X_{\alpha-1}$ by

$$\Psi_{1,3}(x, i) = \begin{cases} y_{\alpha-1} & \text{if } i = 1 \\ x & \text{if } i = 2 \\ x_{\alpha-1} & \text{if } i = 3 \end{cases}$$

This map is G-equivariant and continuous. Using Lemma 3.1 we get $(\Psi_{1,3}(x^{(n)}, i^{(n)}), \Psi_{1,3}(y^{(n)}, j^{(n)})) \in \mathtt{E}_{\alpha-2}(X_{\alpha-1}, T_{\alpha-1})$. By definition of τ_α, there must be n large enough such that $i^{(n)} = 1$ and $j^{(n)} = 3$, therefore we conclude that $(y_{\alpha-1}, x_{\alpha-1}) \in \mathtt{E}_{\alpha-2}(X_{\alpha-1}, T_{\alpha-1})$ thus contradicting (b).

Case 2: α is an odd successor ordinal and $\alpha - 1$ is not a limit ordinal.

(a) Using the embeddings $\phi_i(x) = (x, i)$ for each $i \in I$, we conclude using Lemma 3.1 and $\mathtt{E}_{\alpha-1}(X_{\alpha-1}, T_{\alpha-1}) = X^2_{\alpha-1}$ that

$$\bigcup_{i\in I}(X_{\alpha-1} \times \{i\})^2 \subset \mathtt{E}_{\alpha-1}(X_\alpha, T_\alpha).$$

As $\mathtt{E}_{\alpha-1}(X_\alpha, T_\alpha)$ is transitive and for each $n \in \mathbb{N}$ either $(y_{\alpha-1}, \frac{1}{n}) \sim (y_{\alpha-1}, \frac{1}{n+1})$ or $(x_{\alpha-1}, \frac{1}{n}) \sim (x_{\alpha-1}, \frac{1}{n+1})$ we deduce that $(X_{\alpha-1} \times \{\frac{1}{n} \mid n > 1\})^2 \subset \mathtt{E}_{\alpha-1}(X_\alpha, T_\alpha)$. As this set is dense and $\mathtt{E}_\alpha(X_\alpha, T_\alpha)$ is closed we conclude that $\mathtt{E}_\alpha(X_\alpha, T_\alpha) = X^2_\alpha$.

(b) Suppose $((x_{\alpha-1}, 1), (y_{\alpha-1}, 0)) \in \mathtt{E}_{\alpha-1}(X_\alpha, T_\alpha)$. As $\mathtt{E}_{\alpha-1}(X_\alpha, T_\alpha)$ is transitive, there is a finite sequence of points $\left\{(x^{(k)}, i^{(k)})\right\}_{k=1}^n$ such that $(x^{(1)}, i^{(1)}) = (x_{\alpha-1}, 1)$, $(x^{(n)}, i^{(n)}) = (y_{\alpha-1}, 0)$ and $((x^{(k)}, i^{(k)}), (x^{(k+1)}, i^{(k+1)}))$ is in $\mathtt{E}_{\alpha-2}(X_\alpha, T_\alpha)$. Let $r \in [1, n-1]$ be the largest index such that $i^{(r)} \neq 0$ and $i^{(r+1)} = 0$. Let N be an odd number such that $N > \frac{1}{i^{(r)}}$ and define the map $\Psi_{N,N+2} : X_\alpha \to X_{\alpha-1}$ as

$$\Psi_{N,N+2}(x, i) = \begin{cases} y_{\alpha-1} & \text{if } i \geq \frac{1}{N} \\ x & \text{if } i = \frac{1}{N+1} \\ x_{\alpha-1} & \text{if } i \leq \frac{1}{N+2} \end{cases}$$

As in Case 1, the map $\Psi_{N,N+2} : X_\alpha \to X_{\alpha-1}$ is G-equivariant and continuous, therefore by Lemma 3.1 we obtain that

$$(\Psi_{N,N+2}(x^{(r)}, i^{(r)}), \Psi_{N,N+2}(x^{(r+1)}, i^{(r+1)})) = (y_{\alpha-1}, x_{\alpha-1}) \in \mathtt{E}_{\alpha-2}(X_{\alpha-1}, T_{\alpha-1}).$$

This contradicts the induction hypothesis (b).

Case 3: α is a limit ordinal.

(a) By definition $\mathtt{E}_\alpha(X_\alpha, T_\alpha) = \bigcup_{\beta<\alpha} \mathtt{E}_\beta(X_\alpha, T_\alpha)$. Using Lemma 3.1 with the natural embeddings we obtain that $\bigcup_{n\in\mathbb{N}} X^2_{\alpha(n)} \subset \mathtt{E}_\alpha(X_\alpha, T_\alpha)$. Using that $y_{\alpha(n)} \sim y_{\alpha(n+1)}$ for every $n \in \mathbb{N}$ and that $\mathtt{E}_\alpha(X_\alpha, T_\alpha)$ is an equivalence relation we conclude $\mathtt{E}_\alpha(X_\alpha, T_\alpha) = X^2_\alpha$.

(c) Suppose there exists $n \in \mathbb{N}$ such that $(x_{\alpha(0)}, x_{\alpha(n+1)}) \in \mathtt{E}_{\alpha(n)}(X_\alpha, T_\alpha)$. We define $\phi : X_\alpha \to X_{\alpha(n+1)}$ as follows:

$$\phi(x) = \begin{cases} x & \text{if } x \in X_{\alpha(n+1)} \\ y_{\alpha(n+1)} & \text{otherwise.} \end{cases}$$

One can check that ϕ is G-equivariant and continuous and thus applying Lemma 3.1 we obtain that $\phi((x_{\alpha(0)}, x_{\alpha(n+1)})) = (y_{\alpha(n+1)}, x_{\alpha(n+1)}) \in \mathtt{E}_{\alpha(n)}(X_{\alpha(n+1)}, T_{\alpha(n+1)})$; a contradiction with the induction hypothesis (b).

Case 4: $\alpha - 1$ is a limit ordinal.

(a) The argument follows exactly as in Case 2.

(b) Suppose $((x_{\alpha(0)}, 1), (x_{\alpha(0)}, 0)) \in \mathtt{E}_{\alpha-1}(X_\alpha, T_\alpha)$. Since $\alpha - 1$ is a limit ordinal this implies there exists $n \in \mathbb{N}$ such that $((x_{\alpha(0)}, 1), (x_{\alpha(0)}, 0)) \in \mathtt{E}_{\alpha(n)}(X_\alpha, T_\alpha)$. Choose $m > n$ and define $\varphi_m : X_\alpha \to X_{\alpha-1}$ by

$$\varphi_m(x, i) = \begin{cases} x & \text{if } i < \frac{1}{m} \\ x_{\alpha(m)} & \text{otherwise.} \end{cases}$$

As in the previous cases, φ_m is G-equivariant. As the only identification points are of the form $(x_{\alpha(n)}, \frac{1}{n}) \sim (x_{\alpha(n)}, \frac{1}{n+1})$ we deduce φ_m is continuous. By Lemma 3.1 we obtain that

$$(\varphi_m(x_{\alpha(0)}, 1), \varphi_m(x_{\alpha(0)}, 0)) = (x_{\alpha(m)}, x_{\alpha(0)}) \in \mathtt{E}_{\alpha(n)}(X_{\alpha-1}, T_{\alpha-1})$$

As $m > n$ this contradicts the induction hypothesis (c). □

Theorem 3.4. *For every countable ordinal α the asymptotic class α is non-empty.*

Proof. The previous construction also works in this case. We only need to check the following conditions and the rest of the proof works verbatim.

(i) Let (X, T) and (Y, S) two G-TDS. If $\phi : X \longrightarrow Y$ is a G-equivariant continuous map then for every ordinal $\alpha \geq 0$, we have $\phi(\mathtt{A}_\alpha(X, T)) \subset \mathtt{A}_\alpha(Y, S)$.

(ii) $(X_1, T_1) = (\{0, 1\}^G, \sigma)$ is in the asymptotic class 1.

(iii) (X_2, T_2) is in the asymptotic class 2 and $(x_2, y_2) \notin \mathtt{A}_1(X_2, T_2)$.

Condition (i) is the version of Lemma 3.1 for asymptotic pairs, which holds by Remark 3.2. Conditions (ii) and (iii) are direct. The rest of the proof is done with transfinite induction as in the previous theorem. □

4 Characterization of entropy pairs for TDS with the POTP

We say that a set $H \subset G$ is **P-separated** for $P \subset G$ if for each $d, d' \in H$, $(P \cdot d) \cap (P \cdot d') \neq \varnothing \implies d = d'$.

Lemma 4.1. *Let G be a countable amenable group, $\{F_n\}_{n \in \mathbb{N}}$ a Følner sequence and $J \subset G$ such that*

$$\liminf_{n\to\infty} \frac{|F_n \cap J|}{|F_n|} > 0.$$

For every $P \Subset G$ there exists a subset $H \subset J$ which is P-separated and such that

$$\liminf_{n\to\infty} \frac{|F_n \cap H|}{|F_n|} > 0.$$

Proof. Let H be a maximal P-separated subset of J. Let $C \supset P^{-1} \cdot P$ be a finite subset of G. Every $g \in J$ must satisfy that $H \cap C \cdot g \neq \varnothing$, otherwise the set $H' := H \cup \{g\}$ is P-separated thus contradicting the maximality of H. Therefore, there is a $|C|$-to-1 function from $F_n \cap J$ to $C \cdot F_n \cap H$. This implies that,

$$\frac{|C \cdot F_n \cap H|}{|F_n|} \geq \frac{|F_n \cap J|}{|F_n||C|}.$$

On the other hand, as $\{F_n\}_{n \in \mathbb{N}}$ is Følner, we have,

$$\liminf_{n\to\infty} \frac{|C \cdot F_n \cap H|}{|F_n|} = \liminf_{n\to\infty} \left(\frac{|(C \cdot F_n \setminus F_n) \cap H|}{|F_n|} + \frac{|F_n \cap H|}{|F_n|} \right) = 0 + \liminf_{n\to\infty} \frac{|F_n \cap H|}{|F_n|}.$$

Therefore, we conclude that

$$\liminf_{n\to\infty} \frac{|F_n \cap H|}{|F_n|} \geq \frac{1}{|C|} \liminf_{n\to\infty} \frac{|F_n \cap J|}{|F_n|} > 0.$$

□

The following theorem is a local counterpart of a result by Meyerovitch (Theorem 1.4 of [Mey]) which states that for countable amenable groups, every expansive action with the pseudo-orbit tracing property and positive topological entropy admits non-trivial asymptotic pairs. Although our proof is based on local entropy theory and does not use expansivity, the proof of part (i) has some similar elements to Meyerovitch's proof, which in turn draws some elements from an argument by Schmidt (Proposition 2.1 of [Sch95]) for subshifts of finite type with positive entropy.

Theorem 4.2. *Let (X,T) be a G-TDS with the pseudo-orbit tracing property.*

(i) If $(x,y) \in \mathtt{E}(X,T)$ then $(x,y) \in \overline{\mathtt{A}^{\varepsilon}(X,T)} \setminus \Delta$ for every $\varepsilon > 0$ and there exists an invariant measure μ such that $x, y \in \operatorname{supp}(\mu)$.

(ii) If $(x,y) \in \mathtt{A}(X,T)$ and there exists an invariant measure μ such that $x, y \in \operatorname{supp}(\mu)$ then $(x,y) \in \mathtt{E}(X,T) \cup \Delta$.

Proof. Let $(x,y) \in \mathtt{E}(X,T)$. By Theorem 2.14 we have that $(x,y) \in \mathtt{IE}(X,T)$. Let $\{F_n\}_{n\in\mathbb{N}}$ be a Følner sequence which satisfies the PET.

Part (i). We shall show that for any $\varepsilon > 0$ there exists an ε-asymptotic pair in $B_\varepsilon(x) \times B_\varepsilon(y)$. Let $\delta > 0$ and $S \Subset G$ be respectively the number and finite set given by the POTP for $\varepsilon/2$. Without loss of generality, we assume $S = S^{-1}$. Let $\mathcal{C}$ be a finite δ-cover of X . For every $n \in \mathbb{N}$ we define

$$K_n := \bigvee_{g \in F_n} T^{g^{-1}} \mathcal{C}$$
$$\partial K_n := \bigvee_{g \in S \cdot F_n \setminus F_n} T^{g^{-1}} \mathcal{C}$$

Let $(U_x, U_y) = (B_{\varepsilon/2}(x), B_{\varepsilon/2}(y))$. Since (x,y) is an IE-pair, we know there exists an independence set $J \subset G$ for (U_x, U_y) with positive density $D(J)$ with respect to $\{F_n\}_{n\in\mathbb{N}}$. Using that $D(J) > 0$ we have that for all sufficiently large n

$$|J \cap F_n| \geq \frac{D(J)}{2} \cdot |F_n|.$$

We also have that for all $n \in \mathbb{N}$.

$$|\partial K_n| \leq |\mathcal{C}|^{|S \cdot F_n \setminus F_n|}.$$

Since $\{F_n\}_{n\in\mathbb{N}}$ is a Følner sequence for all sufficiently large m we have

$$2^{\frac{D(J)}{2}|F_m|} > |\mathcal{C}|^{|S \cdot F_m \setminus F_m|} \text{ and } |J \cap F_m| \geq \frac{D(J)}{2} \cdot |F_m|.$$

This implies that for large enough m, $2^{|J \cap F_m|} > |\partial K_m|$. In particular, as J is an independence set for (U_x, U_y), there exist $j \in J \cap F_m$ and $C \in \partial K_m$ and $x', y' \in C$ such that

$$T^j x' \in U_x,$$
$$T^j y' \in U_y,$$

Since $x', y' \in C$, for every $g \in S \cdot F_m \diagdown F_m$ we have that $d(T^g x', T^g y') \leq \delta$. We define the sequence

$$\{z_g\}_{g\in G} \text{ such that } z_g = \begin{cases} T^g x' & \text{if } g \in F_m \\ T^g y' & \text{otherwise.} \end{cases}.$$

We claim that $\{z_g\}_{g\in G}$ is an (S, δ) pseudo-orbit.

If $g \in F_m$ then either $s \cdot g \in F_m$ or $s \cdot g \in S \cdot F_m \setminus F_m$.

- If $s \cdot g \in F_m$ then $z_{s\cdot g} = T^{s\cdot g} x' = T^s(T^g x') = T^s z_g$.
- If $s \cdot g \in S \cdot F_m \setminus F_m$ then $z_{s\cdot g} = T^{s\cdot g} y'$ and $T^s z_g = T^{s\cdot g} x'$. As $d(T^{s\cdot g} x', T^{s\cdot g} y') \leq \delta$ we have $d(z_{s\cdot g}, T^s z_g) \leq \delta$.

If $g \notin F_m$ then either $s \cdot g \notin F_m$ or $s \cdot g \in F_m$.

- If $s \cdot g \notin F_m$ then $z_{s\cdot g} = T^{s\cdot g} y' = T^s(T^g(y')) = T^s z_g$.
- If $s \cdot g \in F_m$, $z_{s\cdot g} = T^{s\cdot g} x'$ and $T^s z_g = T^{s\cdot g} y'$. As $S = S^{-1}$ we have that $g \in S \cdot F_m \setminus F_m$. Therefore $d(T^{s\cdot g} x', T^{s\cdot g} y') \leq \delta$ and thus we have $d(z_{s\cdot g}, T^s z_g) \leq \delta$.

By the POTP there exists $z' \in X$ that $\varepsilon/2$-traces $\{z_g\}_{g\in G}$. We conclude $T^j z' \in B_\varepsilon(x)$ and $T^j y' \in B_\varepsilon(y)$. Also, for every $g \notin F_m$ we have that $d(T^g z', T^g y') \leq \varepsilon/2 + d(z_g, T^g y') = \varepsilon/2$. Therefore (z', y') is ε-asymptotic and thus $(T^j z', T^j y')$ is also ε-asymptotic as we intended to show.

In order to construct the measure having both x, y in its support we use an argument from [KL07]. Let $n \in \mathbb{N}$ and consider an independence set J_n for $(B_{\frac{1}{n}}(x), B_{\frac{1}{n}}(y))$. By independence for any $m \in \mathbb{N}$ sufficiently large there is z_m such that

$$\min\left\{\frac{|\{g \in F_m \mid T^g z_m \in B_{\frac{1}{n}}(x)\}|}{|F_m|}, \frac{|\{g \in F_m \mid T^g z_m \in B_{\frac{1}{n}}(y)\}|}{|F_m|}\right\} > \frac{1}{3} D(J_n).$$

Let ν_n be an accumulation point in the weak-$*$ topology of the sequence $\{\frac{1}{|F_m|}\sum_{g\in F_m} \delta_{T^g z_m}\}_{m\in\mathbb{N}}$. As $\{F_m\}_{m\in\mathbb{N}}$ is a Følner sequence, ν_n is an invariant probability measure that satisfies

$$\nu_n(B_{\frac{1}{n}}(x)) > 0 \text{ and } \nu_n(B_{\frac{1}{n}}(y)) > 0.$$

Consider $\nu = \sum_{n\geq 1} \frac{1}{2^n}\nu_n$. For any neighborhood U_x of x, there is $n \in \mathbb{N}$ such that $B_{\frac{1}{n}}(x) \subset U_x$ and thus $\nu(U_x) \geq \frac{1}{2^n}\nu_n(U_x) > 0$. The same argument holds for y. Therefore $x, y \in \text{supp}(\nu)$.

Part (ii). Let $(x, y) \in \mathtt{A}(X, T)$ and μ an invariant measure such that $x, y \in \text{supp}(\mu)$. Let $\varepsilon > 0$, we want to construct an independence set for $(B_\varepsilon(x), B_\varepsilon(y))$. Let $\delta > 0$ and $S \Subset G$ given by the POTP for $\varepsilon/2$ and without loss of generality, assume $S = S^{-1}$. Let $\delta' \leq \delta$ be such that for any $x', y' \in X$ if $d(x', y') < \delta'$ then for any $s \in S$ $d(T^s x', T^s y') \leq \delta/2$. Since $(x, y) \in \mathtt{A}(X, T)$ there exists $K \Subset G$ such that for $g \notin K$ $d(T^g x, T^g y) \leq \delta/2$. Without loss of generality we can assume that $S \subset K$. We define

$$Q_{\delta'}(x) := \{z \in X : d(T^g x, T^g z) < \delta' \ \forall g \in S \cdot K \diagdown K\}.$$

Note that $Q_{\delta'}(x)$ is a neighborhood of x and therefore $\mu(Q_{\delta'}(x)) > 0$. Since $\{F_n\}_{n\in\mathbb{N}}$ satisfies the PET, we can apply it to the indicator function of $Q_{\delta'}(x)$ and infer the existence of $w \in X$ and a positive density set $J \subset G$ such that $T^j w \in Q_{\delta'}(x)$ for every $j \in J$. Using Lemma 4.1 we can extract $H \subset J$ which is $(S \cdot K)$-separated and has positive density in a subsequence of $\{F_n\}_{n\in\mathbb{N}}$. Let $I \subset H$ be a finite set and $\phi : I \to \{x, y\}$ a function. We define the sequence $\{z_g\}_{g\in G}$ as follows

$$z_g := \begin{cases} T^{g\cdot i^{-1}}\phi(i) & \text{if } g \cdot i^{-1} \in S \cdot K \text{ for some } i \in I \\ T^g w & \text{otherwise} \end{cases}$$

As H is $(S \cdot K)$-separated, if $gi_1^{-1} \in S \cdot K$ and $gi_2^{-1} \in S \cdot K$ then $i_1 = i_2$, meaning that $\{z_g\}_{g\in G}$ is well defined. We claim that $\{z_g\}_{g\in G}$ is an (S, δ) pseudo-orbit. Let $g \in G$ and $s \in S$:

If there is $i \in I$ such that $g \cdot i^{-1} \in S \cdot K$ we have that $z_g = T^{g\cdot i^{-1}}\phi(i)$. There are two cases to consider:

- if $g \cdot i^{-1} \in K$, then $s \cdot g \cdot i^{-1} \in S \cdot K$ and thus $z_{s \cdot g} = T^{s \cdot g \cdot i^{-1}} \phi(i)$. Therefore $d(T^s z_g, z_{s \cdot g}) = 0$.
- if $g \cdot i^{-1} \in S \cdot K \setminus K$ then either $s \cdot g \cdot i^{-1} \in S \cdot K$ and the distance is zero or $z_{s \cdot g} = T^{s \cdot g} w$. In this case, we have

$$d(T^s z_g, z_{s \cdot g}) = d(T^{s \cdot g \cdot i^{-1}} \phi(i), T^{s \cdot g} w) \leq d(T^{s \cdot g \cdot i^{-1}} \phi(i), T^{s \cdot g \cdot i^{-1}} x) + d(T^{s \cdot g \cdot i^{-1}} x, T^{s \cdot g} w).$$

On one hand, we have that $d(T^{s \cdot g \cdot i^{-1}} \phi(i), T^{s \cdot g \cdot i^{-1}} x) \leq \delta/2$ regardless of the value of $\phi(i)$ as $s \cdot g \cdot i^{-1} \notin K$. On the other hand, as $T^i w \in Q_{\delta'}(x)$ and $g \cdot i^{-1} \in S \cdot K \setminus K$ we have that $d(T^{g \cdot i^{-1}} x, T^g w) \leq \delta'$ and therefore $d(T^{s \cdot g \cdot i^{-1}} x, T^{s \cdot g} w) \leq \delta/2$. We conclude that $d(T^s z_g, z_{s \cdot g}) \leq \delta$.

Otherwise, there is no $i \in I$ such that $g \cdot i^{-1} \in S \cdot K$ and thus $z_g = T^g w$. Either the same holds for $s \cdot g$ and the distance is zero or $s \cdot g \cdot i^{-1} \in S \cdot K$ for some $i \in I$. as $S = S^{-1}$, we have that $s \cdot g \cdot i^{-1} \notin K$. Therefore we can use the previous argument to show that $d(T^s z_g, z_{s \cdot g}) \leq \delta$.

By the POTP, there exists $z' \in X$ that $\varepsilon/2$-traces $\{z_g\}_{g \in G}$. Since $S = S^{-1}$ and $S \subset K$, we have that $1_G \in S \cdot K$ and thus $z_i = \phi(i)$. This implies that

$$z' \in \bigcap_{i \in I} T^{i^{-1}} B_\varepsilon(\phi(i)).$$

We conclude H is an independence set for $(B_\varepsilon(x), B_\varepsilon(y))$. □

The following lemma is known (see Lemma 6.2 in [CL14]). We give a proof for completeness.

Lemma 4.3. *Let (X,T) be an expansive G-TDS with expansivity constant ε. Then $\mathtt{A}^{\varepsilon/2}(X,T) = \mathtt{A}(X,T)$.*

Proof. Assume that there exists $(x,y) \in \mathtt{A}^{\varepsilon/2}(X,T) \setminus \mathtt{A}(X,T)$. This implies there exists $\delta > 0$ and an infinite set $\{h_n\}_{n \in \mathbb{N}} \subset G$ such that

$$\delta \leq d(T^{h_n} x, T^{h_n} y) \text{ for all } n \in \mathbb{N}.$$

Let x' and y' be accumulation points of $\{T^{h_n} x\}_{n \in \mathbb{N}}$ and $\{T^{h_n} y\}_{n \in \mathbb{N}}$ respectively. We must have that $x' \neq y'$. Furthermore since $(x,y) \in \mathtt{A}^{\varepsilon/2}(X,T)$ we have that for every sequence of finite sets $\{K_n\}_{n \in \mathbb{N}}$ such that $\cup_{n \in \mathbb{N}} K_n = G$ and any $m \in \mathbb{N}$ for sufficiently large $n \in \mathbb{N}$ we have

$$d(T^{k \cdot h_n} x, T^{k \cdot h_n} y) \leq \varepsilon/2 \text{ for every } k \in K_m.$$

This implies that

$$d(T^g x', T^g y') \leq \varepsilon/2 \text{ for every } g \in K_m.$$

Since this is for any $m \in \mathbb{N}$ we conclude that

$$d(T^g x', T^g y') \leq \varepsilon/2 \text{ for every } g \in G,$$

which contradicts expansivity. □

Putting together Lemma 4.3 and Theorem 4.2 we obtain the following corollaries (in particular we recover Meyerovitch's result [Mey]).

Corollary 4.4. *Let (X,T) be an expansive G-TDS with POTP and positive topological entropy. Then there exists a non-trivial asymptotic pair.*

Proof. By Theorem 2.4 any G-TDS with positive topological entropy admits an entropy pair (x,y). By Theorem 4.2 and Lemma 4.3 we get that $(x,y) \in \overline{\mathtt{A}(X,T)} \setminus \Delta$. □

We remind the reader that $\mathcal{S}(X,T)$ is the set of all $x \in X$ for which there exists an invariant measure μ for which $x \in \operatorname{supp}(\mu)$ and $\Delta_{\mathcal{S}(X,T)}$ is the diagonal of $\mathcal{S}(X,T)$.

Corollary 4.5. *Let (X,T) be an expansive G-TDS with POTP. Then*

$$\overline{\mathtt{A}(X,T) \cap (\mathcal{S}(X,T))^2} \subset \mathtt{E}(X,T) \cup \Delta \subset \overline{\mathtt{A}(X,T)} \cap (\mathcal{S}(X,T))^2$$

Proof. By the first part of Theorem 4.2 and Lemma 4.3 we have that $\mathtt{E}(X,T)\cup\Delta \subset \overline{\mathtt{A}(X,T)}\cap(\mathcal{S}(X,T))^2$. By the second part we obtain that $\overline{\mathtt{A}(X,T)\cap(\mathcal{S}(X,T))^2} \subset \mathtt{E}(X,T)\cup\Delta$. As $\mathtt{E}(X,T)\cup\Delta$ is closed we get $\overline{\mathtt{A}(X,T)\cap(\mathcal{S}(X,T))^2} \subset \mathtt{E}(X,T)\cup\Delta$. □

In general, if (X,T) does not admit a fully supported measure the inclusions can be strict.

Example 4.6. Let $X \subset \{◀, 0, 1, ▶\}^{\mathbb{Z}}$ be the set of all configurations where the patterns $\{▶ 0, ▶ 1, ▶◀, 0 ◀, 1 ◀, 01, 10\}$ do not appear. The subshift (X,σ) is an expansive $\mathbb{Z}$-TDS with the POTP. It is easy to see that $\mathcal{S}(X,T) = \{◀^{\mathbb{Z}}, 0^{\mathbb{Z}}, 1^{\mathbb{Z}}, ▶^{\mathbb{Z}}\}$ and thus that $\overline{\mathtt{A}(X,T)\cap(\mathcal{S}(X,T))^2} = \Delta_{\mathcal{S}(X,T)}$. On the other hand we have $(\mathcal{S}(X,T))^2 \subset \overline{\mathtt{A}(X,T)}$ and thus $\overline{\mathtt{A}(X,T)}\cap(\mathcal{S}(X,T))^2 = (\mathcal{S}(X,T))^2$.

Corollary 4.7. *Let (X,T) be an expansive G-TDS with POTP which admits a fully supported invariant measure. Then for every ordinal $\alpha > 0$ we have $\mathtt{E}_\alpha(X,T) = \mathtt{A}_\alpha(X,T)$.*

Proof. If (X,T) admits a fully supported invariant measure then $\mathcal{S}(X,T) = X$. Using Corollary 4.5 we obtain that $\overline{\mathtt{A}(X,T)} = \mathtt{E}(X,T)\cup\Delta$. By definition of both hierarchies we deduce that $\mathtt{E}_\alpha(X,T) = \mathtt{A}_\alpha(X,T)$ for every ordinal $\alpha > 0$. □

5 The asymptotic hierarchy for subshifts

A particularly interesting class of expansive G-TDSs are subshifts. Given a finite alphabet $\mathcal{A}$ and $X \subset \mathcal{A}^G$ we say X is a **subshift** if it is closed under the product topology and invariant under the left shift action σ defined by $\sigma^g(x)_h = x_{g^{-1}h}$. We denote by $L(X)$ the **language** of X and for a $F \Subset G$ we write $L_F(X) = \mathcal{A}^F \cap L(X)$ for the set of patterns appearing in X with support F. We say that a subshift is of **finite type (SFT)** if it is equal to the complement of the shift-closure of a finite union of cylinders. We say it is a **sofic** subshift if it is a factor of an SFT, that is, the image of an SFT under a G-equivariant continuous map.

In the case where X is a subshift one can characterize asymptotic pairs as follows: $(x,y) \in X^2$ is an asymptotic pair if and only if there exists $F \Subset G$ such that $x|_{G\setminus F} = y|_{G\setminus F}$.

Example 5.1. The sunny side up subshift $X_{\leq 1}$ is in the asymptotic class 0.

$$X_{\leq 1} = \{x \in \{0,1\}^{\mathbb{Z}} \mid 1 \in \{x_n, x_m\} \implies n = m\}.$$

A subshift in the asymptotic class 0 is extremely simple. It consists of a single asymptotic class and thus it is either finite or countable and contains a unique uniform configuration. In particular, the only SFT with a fully supported measure satisfying this property is the one consisting of a unique uniform configuration.

As the following example shows, constructing examples in the asymptotic class 1 is quite simple.

Example 5.2. Let $X = \{0,1\}^G$ be the full 2-shift. The fixed points 0^G and 1^G are obviously not asymptotic, but it is easy to see that $\mathtt{A}(X,T)$ is dense in X^2.

In what follows, we will describe the second and third level of the asymptotic hierarchy in the case where X is a subshift. For that, we will need a few definitions.

Definition 5.3. *Let $X \subset \mathcal{A}^G$ be a subshift, $F \Subset G$ and $p,q \in L_F(X)$. We say p,q are* ***exchangeable*** *if there exists an asymptotic pair (x,y) such that $x|_F = p$ and $y|_F = q$.*

Proposition 5.4. *A subshift X has asymptotic class at most 1 if and only if for every $F \Subset G$ every pair $p,q \in L_F(X)$ is exchangeable.*

Proof. Suppose X has asymptotic class at most 1, that is, $\overline{\mathtt{A}(X,\sigma)} = X^2$. Given $p,q \in L_F(X)$ choose $(x,y) \in X^2$ such that $x|_F = p$ and $y|_F = q$. By definition there is $\{(x^n,y^n)\}_{n\in\mathbb{N}} \subset \mathtt{A}(X,\sigma)$ converging to (x,y). Choosing a sufficiently large n such that $x^n|_F = x|_F = p$ and $y^n|_F = y|_F = q$ gives an asymptotic pair for p,q, therefore they are exchangeable. Conversely, Let $\{F_n\}_{n\in\mathbb{N}} \nearrow G$ be an increasing sequence of finite subsets of G and for $(x,y) \in X^2$ let $p_n := x|_{F_n}$ and $q_n := y|_{F_n}$. As every pair of patterns is exchangeable, there is an asymptotic pair $(x^n,y^n) \in \mathtt{A}(X,\sigma)$ for (p_n,q_n). As $\{F_n\}_{n\in\mathbb{N}} \nearrow G$ we have that (x^n,y^n) converges to (x,y) as n goes to infinity, therefore $(x,y) \in \overline{\mathtt{A}(X,\sigma)}$. □

Definition 5.5. *Let $F \Subset G$ and $p, q \in \mathcal{A}^F$ two patterns. We say p, q are n-**chain exchangeable** if there exists a sequence of patterns $r_0, r_1, \dots, r_n \in \mathcal{A}^F$ such that $p = r_0$, $q = r_n$ and r_{i-1}, r_i are exchangeable for every $i \in \{1, \dots, n\}$. We say that p, q are **chain exchangeable** if they are n-chain exchangeable for some $n \in \mathbb{N}$.*

Definition 5.6. *We say that a subshift X has **chain exchangeability (CE)** if every pair of patterns in $L(X)$ over the same support is chain exchangeable. Furthermore, We say that X has **bounded chain exchangeability (BCE)** if there exists a uniform constant $N \in \mathbb{N}$ such that every pair of patterns in $L(X)$ over the same support is N-chain exchangeable.*

Theorem 5.7 ([Pav13],[Pav18])**.** *Let X be a $\mathbb{Z}^d$-SFT admitting a fully supported invariant measure.*

(i) X has chain exchangeability if and only if every non-trivial zero-dimensional factor of X has positive topological entropy.

(ii) If X has bounded chain exchangeability then it has topological CPE.

In [Pav18] Pavlov asked whether every $\mathbb{Z}^d$-SFT that has topological CPE must satisfy bounded chain exchangeability. Theorem 6.22 answers that question negatively.

Proposition 5.8. *If a G-subshift X has bounded chain exchangeability then $\mathtt{A}_2(X, \sigma) = X^2$. That is, it is either in the asymptotic class* 0, 1 *or* 2*.*

Proof. Suppose X has BCE and let N be the associated constant. Given $(x, y) \in X^2$ consider an increasing sequence $\{F_n\}_{n\in\mathbb{N}} \nearrow G$ of finite subsets of G and the patterns $p_n = x|_{F_n}$ and $q_n = y|_{F_n}$. By BCE, there are patterns $r^n_0, \dots r^n_N$ such that:

$$p_n = r^n_0, q_n = r^n_N \text{ and } r_i, r_{i+1} \text{ are exchangable.}$$

We can therefore find configurations $z^{(n,0)}, \dots, z^{(n,N-1)}$ and $\tilde{z}^{(n,1)}, \dots, \tilde{z}^{(n,N)}$ such that $(z^{(n,i)}, \tilde{z}^{(n,i+1)}) \in \mathtt{A}(X, \sigma)$ and $z^{(n,0)}|_{F_n} = r^n_0$, $\tilde{z}^{(n,N)}|_{F_n} = r^n_N$ and for $i \in \{1, \dots, N-1\}$ $z^{(n,i)}|_{F_n} = \tilde{z}^{(n,i)}|_{F_n} = r^n_i$. By sequential compactness of X^{2N-2}, the sequence $\{(z^{(n,0)}, \dots, z^{(n,N-1)}, \tilde{z}^{(n,1)}, \dots, \tilde{z}^{(n,N)})\}_{n\in\mathbb{N}}$ admits a subsequence which converges to some $(z^0, \dots, z^{N-1}, \tilde{z}^1, \dots, \tilde{z}^N) \in X^{2N-1}$. By definition of the sequence and the fact that we chose $\{F_n\}_{n\in\mathbb{N}} \nearrow G$, it is clear that $x = z^0$ and $y = \tilde{z}^N$ and for $i \in \{1, \dots, N-1\}$ $z^i = \tilde{z}^i$. Moreover, as $(z^{(n,i)}, \tilde{z}^{(n,i+1)}) \in \mathtt{A}(X, \sigma)$ we have that $(z^i, \tilde{z}^{i+1}) \in \overline{\mathtt{A}(X, \sigma)}$. Combining these two facts we obtain that $(x, y) \in (\overline{\mathtt{A}(X, \sigma)})^N \subset (\overline{\mathtt{A}(X, \sigma)})^+$. As (x, y) was arbitrary and $(\overline{\mathtt{A}(X, \sigma)})^+ = \mathtt{A}_2(X, \sigma)$ we conclude $\mathtt{A}_2(X, \sigma) = X^2$. □

Using Theorem 4.2 and Proposition 5.8 we generalize Theorem 5.7 part (ii) for arbitrary countable amenable groups.

Corollary 5.9. *Let X be a G-SFT admitting a fully supported measure. If X has bounded chain exchangeability, then it has topological CPE.*

Question 5.10. *Is there a G-SFT in the asymptotic class 2 which does not satisfy bounded chain exchangability?*

The answer to the previous question is negative for topologically weakly mixing subshifts, even if they are not SFTs. Recall that a dynamical system (X, T) is **topologically weakly mixing** if $(X \times X, T \times T)$ is transitive.

Proposition 5.11. *A topologically weakly mixing G-subshift is in the asymptotic class* $0, 1$ *or* 2 *if and only if it has bounded chain exchangeability.*

Proof. Suppose X is not bounded chain exchangeable, then for every $n \in \mathbb{N}$ there is a support F_n and patterns (p_n, q_n) which are not n-chain exchangeable. As X^2 is irreducible we have that there is $g_1 \in G$ such that $[p_0] \times [q_0] \cap T^{g_1}([p_1] \times [q_1]) \neq \emptyset$. Let g_0 be the identity of G and iterate this argument to obtain a sequence $\{g_i\}_{i\in\mathbb{N}}$ such that for every $m \in \mathbb{N}$ the set $U_m = \bigcap_{i\leq m} T^{g_i}([p_i] \times [q_i])$ is non-empty. Choose $(x^m, y^m) \in U_m$ and extract an accumulation point $(x, y) \in X^2$. As X has class at most two, there must be $N \in \mathbb{N}$ and configurations $z^0, \dots, z^N$ such that $z^0 = x$, $z^N = y$ and $(z^i, z^{i+1}) \in \overline{\mathtt{A}(X, \sigma)}$. By definition of convergence, we have that for each $m \in \mathbb{N}$ $\sigma^{g_m^{-1}}(x)|_{F_m} = p_m$ and $\sigma^{g_m^{-1}}(y)|_{F_m} = q_m$. Let $m > N$ and $r_i := \sigma^{g_m^{-1}}(z^i)|_{F_m}$. We have that $p_m = r_0$, $q_m = r_N$ and (r_i, r_{i+1}) is exchangeable. Hence we have a chain of length $N < m$ for the pair (p_m, q_m) which is not m-chain-exchangeable by assumption, thus yielding a contradiction. The other direction is given by Proposition 5.8. □

Theorem 5.12 ([Pav13])**.** *There exists a $\mathbb{Z}^2$-SFT which admits a fully supported measure and has bounded chain exchangeability for which there are non-exchangeable patterns.*

Corollary 5.13. *There exists a $\mathbb{Z}^2$-SFT in the CPE class 2.*

Proof. Using Proposition 5.8 and Proposition 5.4 we obtain that Pavlov's example is in the asymptotic class 2. Also, since it admits a fully supported measure Theorem 4.2 yields the result. □

6 A $\mathbb{Z}^3$-SFT in the CPE class 3

From this point forward, we will use the letters x, y, z exclusively to represent coordinates in $\mathbb{Z}^3$. Before presenting our main example we construct a sofic $\mathbb{Z}^2$-subshift in the asymptotic class 3. The purpose of this construction is to illustrate the main ideas of the proof of Theorem 6.22 in a simpler setting.

First we construct a $\mathbb{Z}^2$-SFT whose alphabet is given by the tiles in Figure 3. We will refer to the first tile as a **white tile** and denote it by □, the second and third as **line tiles** and the rest as **corner tiles**. The adjacency rules are those of Wang tilings, namely, two tiles can be adjacent to each other if and only if the colors match along their shared border, see Figure 4. We denote this SFT as X.

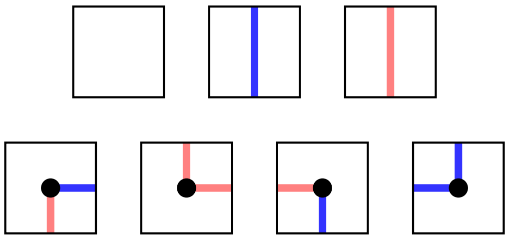


Figure 3: The alphabet.

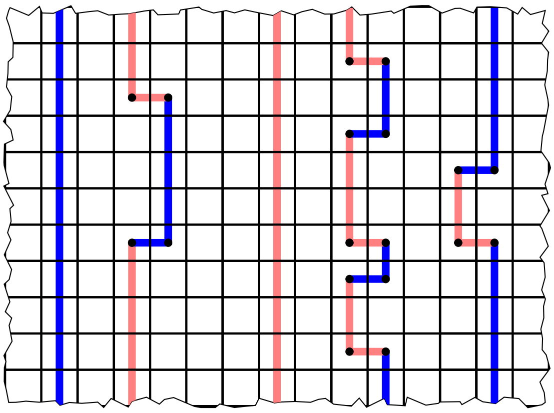


Figure 4: A configuration in X.

Let X_W be the symbolic factor of X obtained by removing the colors in every configuration. We will call this example the **Worm Shift**. By definition, this is a sofic $\mathbb{Z}^2$-subshift. See Figure 5.

After proving a series of claims we will show that the Worm Shift is in the asymptotic class 3.

Definition 6.1. *Given $c \in X_W$ we say $W \subset \mathbb{Z}^2$ is a **worm** if*

1. *There exists $x \in \mathbb{Z}$ such that $W \subset \{x, x+1\} \times \mathbb{Z}$. Furthermore if W is contained in $\{x\} \times \mathbb{Z}$ or $\{x+1\} \times \mathbb{Z}$ we say it is a **straight worm.***
2. *W is infinite and 4-connected.*

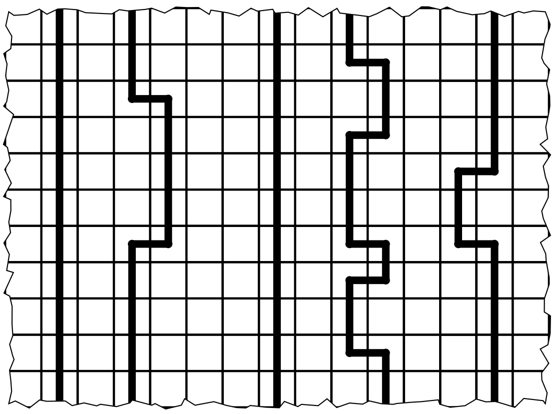
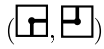


Figure 5: A configuration in X_W.

3. $c_{(x,y)} \neq \square$ *for every* $(x,y) \in W$.

4. If $(x,y),(x+1,y) \in W$ *then* $(c_{(x,y)}, c_{(x+1,y)}) =$  *or* $(c_{(x,y)}, c_{(x+1,y)}) =$ .

Given the constraints of the SFT X it is not hard to check the following remarks.

Remark 6.2. Given $c \in X_W$ and $c_{(x,y)} \neq \square$ there exists a worm W such that $(x,y) \in W$.

Remark 6.3. If $c \in X_W$ and $(c, \square^{\mathbb{Z}^2}) \in \mathtt{A}(X_W, \sigma)$ then $c = \square^{\mathbb{Z}^2}$.

We begin by showing that $\mathtt{A}_2(X_W,\sigma) \neq X_W^2$.

Claim 6.4. *For each* $c \in X_W$ *with* $c \neq \square^{\mathbb{Z}^2}$ *we have* $(c, \square^{\mathbb{Z}^2}) \notin \mathtt{A}_1(X_W,\sigma)$.

Proof. Assume that $(c, \square^{\mathbb{Z}^2}) \in \mathtt{A}_1(X_W,\sigma)$ and there exists $(x,y) \in \mathbb{Z}^2$ such that $c_{(x,y)} \neq \square$. There exists $c', d \in X_W$ such that $(c', d) \in \mathtt{A}(X_W,\sigma)$ with $c'|_{(x,y)+[-2,2]^2} = c|_{(x,y)+[-2,2]^2}$ and $d|_{(x,y)+[-2,2]^2} = \square^{\mathbb{Z}^2}|_{(x,y)+[-2,2]^2}$. This implies that $c'_{(x,y)} \neq \square$ and hence by the previous remarks there exists a worm (of c') contained in $\{x-1, x, x+1\} \times \mathbb{Z}$. Since $(c', d) \in \mathtt{A}(X_W,\sigma)$ we conclude that there exists a worm (of d) contained in $\{x + [-2,2]\} \times \mathbb{Z}$. We obtain a contradiction since we already noted that $d|_{(x,y)+[-2,2]^2} = \square^{\mathbb{Z}^2}|_{(x,y)+[-2,2]^2}$. □

Claim 6.5. *For each* $c \in X_W$ *with* $c \neq \square^{\mathbb{Z}^2}$ *we have* $(c, \square^{\mathbb{Z}^2}) \notin \mathtt{A}_2(X_W,\sigma)$.

Proof. Assume that $(c, \square^{\mathbb{Z}^2}) \in \mathtt{A}_2(X_W,\sigma)$. This implies there exists a chain $c = c^0, \dots, c^n = \square^{\mathbb{Z}^2}$ in X_W such that $(c^k, c^{k+1}) \in \mathtt{A}_1(X_W,\sigma)$. By Claim 6.4 every element of the chain must be $\square^{\mathbb{Z}^2}$, therefore $c = \square^{\mathbb{Z}^2}$. □

It remains to show that $\mathtt{A}_3(X_W,\sigma) = X_W^2$. We will do it first in simple cases and build up to a general pair using the previous steps.

Claim 6.6. $\mathtt{A}_3(X_W,\sigma) = X_W^2$.

Proof. We proceed through five steps. The goal of the first four steps is to show that any pair of configurations having the same finite number of worms must belong to $\mathtt{A}_2(X_W,\sigma)$. The fifth step shows that those pairs of configurations are dense.

Step 1 Two configurations, c and d, with one straight worm each, in consecutive positions.

That is, there exists $n \in \mathbb{Z}$ such that $c_{(x,y)} = \square$ if and only if $x \neq n$ and $d_{(x,y)} = \square$ if and only if $x \neq n+1$. Let $m \geq 1$ and $H := \{(x,y) \in \mathbb{Z}^2 : |y| > m, x \neq n\} \cup \{(x,y) \in \mathbb{Z}^2 : |y| = m, x \neq n, x \neq n+1\} \cup \{(x,y) \in \mathbb{Z}^2 : |y| < m, x \neq n+1\}$. We define $e^{(m)} \in X_W$ as the only configuration that satisfies

$e^{(m)}_{(x,y)} = \square$ if and only if $(x, y) \in H$. Note that $(e^{(m)}, c) \in \mathtt{A}(X_W, \sigma)$ and $e^{(m)}$ converges to d. This implies that $(c, d) \in \mathtt{A}_1(X_W, \sigma)$.

Step 2 Two configurations c and d, with one straight worm each.

In this case, there exists $n, n' \in \mathbb{Z}$ such that $c_{(x,y)} = \square$ if and only if $x \neq n$ and $d_{(x,y)} = \square$ if and only if $x \neq n'$. Without loss of generality assume that $n < n'$. For $k \in [n, n']$ we define c^k as: $c^k_{(x,y)} = \square$ if and only if $x \neq k$. Using step 1 we have that $(c^i, c^{i+1}) \in \mathtt{A}_1(X_W, \sigma)$. Since $c^n = c$ and $c^{n'} = d$ we conclude that $(c, d) \in \mathtt{A}_2(X_W, \sigma)$.

Step 3 Two configurations c and d, with one worm each (not necessarily straight).

Let W_c and W_d be the worms of c and d respectively. There exists $i, j \in \mathbb{Z}$ such that $W_c \subset \{i, i+1\} \times \mathbb{Z}$ and $W_d \subset \{j, j+1\} \times \mathbb{Z}$. We define $c', d' \in X_W$ as follows: $c'_{(x,y)} = \square$ if and only if $y \neq i$ and $d'_{(x,y)} = \square$ if and only if $y \neq j$. Using a similar argument as in step 1 we obtain that $(c, c') \in \mathtt{A}_1(X_W, \sigma)$ and $(d, d') \in \mathtt{A}_1(X_W, \sigma)$. Using step 2 we have $(c', d') \in \mathtt{A}_2(X_W, \sigma)$. As $\mathtt{A}_2(X_W, \sigma)$ is an equivalence relation we conclude $(c, d) \in \mathtt{A}_2(X_W, \sigma)$.

Step 4 Two configurations c and d, with exactly m worms each.

As both configurations have finitely many worms, there exists a value $n \in \mathbb{Z}$ such that any worm in either c or d is contained in $\{-n, \dots, n\} \times \mathbb{Z}$. Let e be the configuration containing m straight worms at positions $n+2, n+4, \dots, n+2m$. Using the tools in Step 2 and Step 3 we can move the rightmost worm in c towards the right one step at a time until it becomes a straight worm at $n+2m$. Iterating this procedure for all worms in c from right to left we obtain that $(c, e) \in \mathtt{A}_2(X_W, \sigma)$. Similarly, $(d, e) \in \mathtt{A}_2(X_W, \sigma)$ and thus $(c, d) \in \mathtt{A}_2(X_W, \sigma)$.

Step 5 Two arbitrary configurations c and d.

Let $Y := \big\{(c', d') \in X_W^2 :\ c'$ and d' have the same finite number of worms$\big\}$. We will see that Y is dense in X_W^2. Let $n \in N$ and consider $[-n, n]^2 \subset \mathbb{Z}^2$. Let $N_n(c)$ be the number of worms for c contained in $[-n-1, n+1]^2 \times \mathbb{Z}$.

If $N_n(c) = N_n(d)$ then construct c' and d' as the configurations with $N_n(c)$ worms in same positions as c and d. Otherwise without loss of generality $N_n(c) < N_n(d)$. We construct d' exactly as before and construct c' exactly as before in the window $[-n-1, n+1]^2 \times \mathbb{Z}$ and then add $N_n(d) - N_n(c)$ worms outside this window. In both cases $(c', d') \in Y$ coincide with (c, d). in $[-n, n]^2$. As n is arbitrary this shows that Y is dense in X_W^2.

By step 4 we have $Y \subset \mathtt{A}_2(X_W, \sigma)$. As $\mathtt{A}_3(X_W, \sigma)$ is closed we conclude $(c, d) \in \mathtt{A}_3(X_W, \sigma)$. $\square$

Proposition 6.7. *The Worm Shift is in the asymptotic class* 3.

Proof of Proposition 6.7. By Claim 6.5 we have $\mathtt{A}_2(X_W, \sigma) \neq X_W^2$ and by Claim 6.6 we have $\mathtt{A}_3(X_W, \sigma) = X_W^2$. $\square$

The Worm Shift has dense strongly periodic configurations and therefore admits a fully supported measure. However, X_W is not an SFT. The next construction uses similar ideas to the previous one but does indeed yield an SFT.

Theorem 6.8. *There is a* $\mathbb{Z}^3$*-SFT in the asymptotic class* 3.

We begin by defining a $\mathbb{Z}^2$-SFT X_{struct} on the alphabet Σ given by all the tiles that can be obtained by coloring the thick black lines of the squares seen on Figure 6 with the colors blue and red (in all subsequent figures red is portrayed with a light tone and blue in a dark tone to facilitate the lecture in grayscale). The adjacency rules are those of Wang tilings, namely, two tiles can be adjacent to each other if and only if the lines and colors match along their shared border.

The first three squares are special and are called a **white tile**, a **diagonal tile** and a **dash tile** respectively. We refer to the rest of the squares as **wire tiles**. For a wire tile t we define the values $\mathsf{SW}(t), \mathsf{SE}(t), \mathsf{NW}(t), \mathsf{NE}(t)$ as 0 if the color which is closest to the southwest, southeast, northwest and

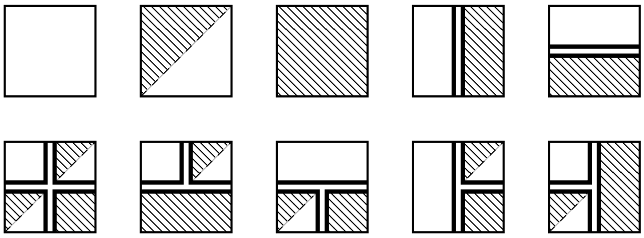
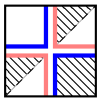
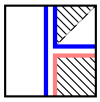
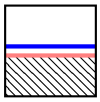


Figure 6: The alphabet, up to coloring of the wires.

northeast corners of the square t is blue and as 1 if it is red. For example, the values associated to the following tiles are:

$t_1 =$  has $\mathsf{NW}(t_1) = 0$ $\mathsf{NE}(t_1) = 1$ $\mathsf{SW}(t_1) = 1$ $\mathsf{SE}(t_1) = 0$

$t_2 =$  has $\mathsf{NW}(t_1) = 0$ $\mathsf{NE}(t_1) = 0$ $\mathsf{SW}(t_1) = 0$ $\mathsf{SE}(t_1) = 1$

$t_3 =$  has $\mathsf{NW}(t_1) = 0$ $\mathsf{NE}(t_1) = 0$ $\mathsf{SW}(t_1) = 1$ $\mathsf{SE}(t_1) = 1$

The global structure of a configuration in X_{struct} is that of a partition of $\mathbb{Z}^2$ into squares with colored contours. Note that besides from the special case of configurations which do not contain wire tiles, the contour structure gives a unique way to color the associated partition into blue and red zones. On Figure 7 we show an example of an admissible local configuration.

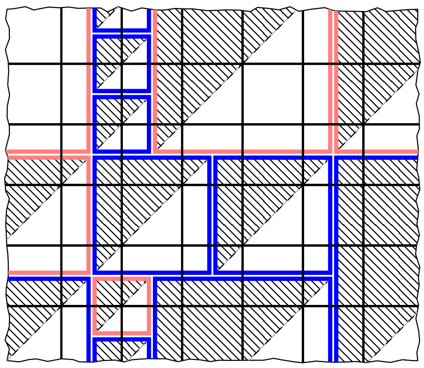
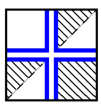


Figure 7: A valid pattern in X_{struct}. The contours of each region determine blue and red "zones"

Before introducing our $\mathbb{Z}^3$ example, we need the following claim.

Claim 6.9. *For any $n \geq 1$ and $p \in L_{[-n,n]^2}(X_{struct})$ there exists $c \in X_{struct}$ such that:*

1. $c|_{[-n,n]^2} = p$.
2. *For every $u \in \mathbb{Z}^2$ such that $||u||_\infty > 4n$ we have*

$c_u =$ 

Proof. Any partial square appearing in p can be extended to a complete square. More precisely, if the largest length of a side of a partial square is k, the completed region can be constructed with a border length of at most $2k$ (the worst case given by a NW or SE corner of a square with only greys or whites inside respectively). As the length of a side of a partial square appearing in p is at most $2n+1$, one can always find such a completion inside the support $[-4n, 4n]$. Let $\widetilde{p}$ be a pattern obtained after completing all of the partial squares appearing in p. One can choose the color of the wires bounding $\widetilde{p}$ to be only blue, and put an outgoing wire everywhere. This pattern can be extended to a $\mathbb{Z}^2$ configuration as demanded. This procedure is illustrated in Figure 8. □

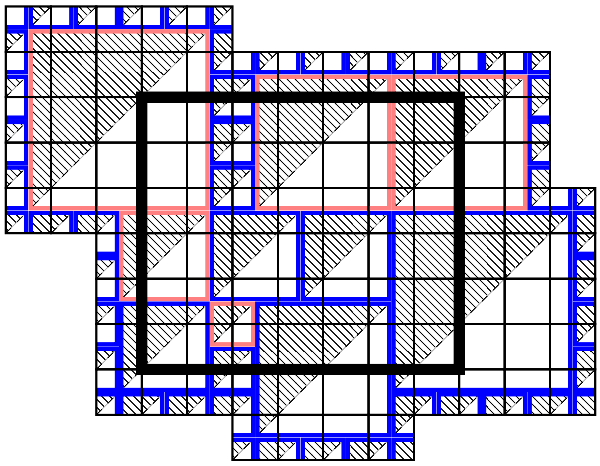


Figure 8: Completion of the pattern From Figure 7 to a pattern which can be extended using only the blue cross wire tile.

Consider now the alphabet $C = \{$■,□$\}$ of filled cubes and empty cubes. We define the **Good Wave Shift** $X_{\texttt{GW}}$ as the $\mathbb{Z}^3$-SFT $X_{\texttt{GW}}$ over the alphabet $\Sigma \times C$ by the following rules.

Given $c \in (\Sigma \times C)^{\mathbb{Z}^3}$ we denote by $\pi_1(c)$ and $\pi_2(c)$ its projection to the first and second coordinate respectively. For $k \in \mathbb{Z}$ we define the z-plane at k as the set $\mathbb{Z}^2 \times \{k\} \subset \mathbb{Z}^3$. If we do not wish to specify k we will just refer to a z-plane.

1. (Σ-Structure I) For each $k \in \mathbb{Z}$ and $c \in X_{\texttt{GW}}$, $\pi_1(c)$ must contain a valid configuration of X_{struct} in the z-plane at k.

2. (Σ-Structure II) If a non-white tile from Σ appears in position (x, y, z), then the white tile must appear at $(x, y, z+1)$ and $(x, y, z+2)$.

3. (C-Structure I) If a ■ appears at coordinate (x, y, z), then a □ must appear at $(x, y, z+1)$ and $(x, y, z+2)$.

4. (C-Structure II) If a ■ appears at coordinate (x, y, z), then:
   - Exactly one ■ appears in either $(x+1, y, z-1)$, $(x+1, y, z)$ or $(x+1, y, z+1)$.
   - Exactly one ■ appears in either $(x-1, y, z-1)$, $(x-1, y, z)$ or $(x-1, y, z+1)$.
   - Exactly one ■ appears in either $(x, y+1, z-1)$, $(x, y+1, z)$ or $(x, y+1, z+1)$.
   - Exactly one ■ appears in either $(x, y-1, z-1)$, $(x, y-1, z)$ or $(x, y-1, z+1)$.

5. ($\Sigma \times C$-Structure I) If a non-white tile appears at position $(x, y, z) \in \mathbb{Z}^3$, then a ■ must appear either at (x, y, z) or $(x, y, z+1)$.

6. ($\Sigma \times C$-Structure II) If a wire tile $t \in \Sigma$ appears at position $(x, y, z) \in \mathbb{Z}^3$, then:
   - ■ appears at $(x, y, z + SW(t))$.
   - ■ appears at $(x+1, y, z + SE(t))$.
   - ■ appears at $(x, y+1, z + NW(t))$.
   - ■ appears at $(x+1, y+1, z + NE(t))$.

7. ($\Sigma \times C$-Structure III) If a  appears at (x, y, z), then:

- If  appears at $(x+1, y, z+1)$, then there is a wire tile t appearing at (x, y, z) with $SW(t) = 0$ and $SE(t) = 1$.
- If  appears at $(x+1, y, z-1)$, then there is a wire tile t appearing at $(x, y, z-1)$ with $SW(t) = 1$ and $SE(t) = 0$.
- If  appears at $(x, y+1, z+1)$, then there is a wire tile t appearing at (x, y, z) with $SW(t) = 0$ and $NW(t) = 1$.
- If  appear sat $(x, y+1, z-1)$, then there is a wire tile t appearing at $(x, y, z-1)$ with $SW(t) = 1$ and $NW(t) = 0$.

Clearly all of these rules can be codified with a finite amount of forbidden patterns, therefore $X_{\texttt{GW}}$ is a $\mathbb{Z}^3$-SFT. This concludes the construction, we will now prove a series of claims that are stepping stones towards Theorem 6.22.

Claim 6.10. *Let $c \in X_{\texttt{GW}}$ and suppose $\pi_1(c)_{(x,y,z)} \neq \square$. Then for every $(i, j) \in \mathbb{Z}^2$ $\pi_1(c)_{(i,j,z+1)} = \pi_1(c)_{(i,j,z+2)} = \square$.*

Proof. Let us define a map $\gamma : X_{\text{struct}} \to \{\square, \blacksquare\}^{\mathbb{Z}^2}$ by declaring for $d \in X_{\text{struct}}$ that $\gamma(d)_{(x,y)} = \square$ if and only if (x, y) belongs to an unbounded 4-connected component of $d^{-1}(\square)$. That is, there is an unbounded path $v \colon \mathbb{N} \to \mathbb{Z}^2$ such that $v(0) = (x, y)$ and $d_{v(n)}$ is the white tile for every $n \in \mathbb{N}$.

Every unbounded 4-connected component of white tiles can be obtained as a limit of bounded right triangles in which one or more of the sides are sent to infinity. Therefore there are, up to translation, finitely many possible shapes for a maximal unbounded 4-connected component of white tiles appearing in a configuration in X_{struct}. A section of all seven of these possibilities are shown in Figure 9. Therefore, the configurations of $\{\square, \blacksquare\}^{\mathbb{Z}^2}$ that can be obtained as images of configurations in X_{struct} under γ are those where the white tiles are covering disjoint and 4-disconnected unions of the white unbounded components shown in Figure 9. Note that at most three unbounded components may appear in such a configuration, we illustrate an example with three unbounded components in Figure 10.

Suppose that e, e' are two configurations in $\gamma(X_{\text{struct}})$ satisfying that for each $(x, y) \in \mathbb{Z}^2$ if $e_{(x,y)} = \blacksquare$ then $e'_{(x,y)} = \square$. It is not hard to prove, using the description of configurations in $\gamma(X_{\text{struct}})$ given above, that either $e = \square^{\mathbb{Z}^2}$ or $e' = \square^{\mathbb{Z}^2}$.

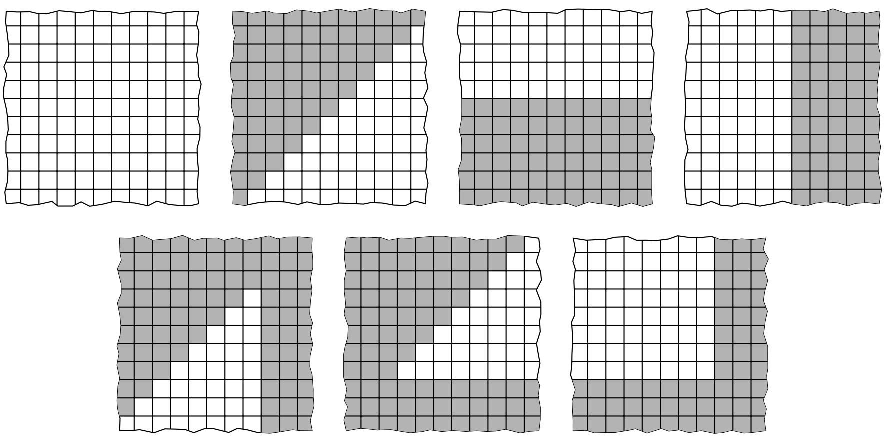


Figure 9: Every unbounded configuration of white tiles in X_{struct} must correspond to one of the above shapes up to translation.

By rule (Σ-Structure II) we know that any non-white tile appearing in $\pi_1(c)$ at $(x', y', z') \in \mathbb{Z}^3$ forces a white tile in $\pi_1(c)$ at $(x', y', z'+1)$ and $(x', y', z'+2)$. It is also easy to verify that, in X_{struct}, if a region is bounded by only white tiles, then it can only contain the white tiles inside the region. Putting these two facts together we obtain that not only non-white tiles force white tiles in the next two

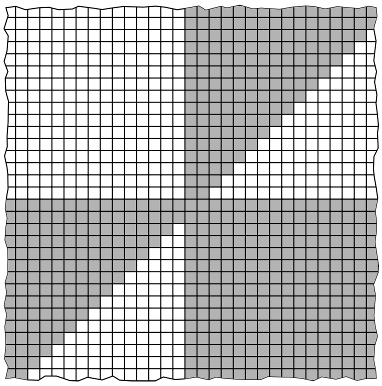


Figure 10: A possible configuration obtained through γ.

coordinates, but also white tiles appearing in bounded regions. Formally, if $\gamma(\pi_1(c)|_{\mathbb{Z}^2\times\{z'\}})_{(x',y')} = \blacksquare$ then $\gamma(\pi_1(c)|_{\mathbb{Z}^2\times\{z'+1\}})_{(x',y')} = \square$ and $\gamma(\pi_1(c)|_{\mathbb{Z}^2\times\{z'+2\}})_{(x',y')} = \square$.

Let $i \in \{0,1,2\}$, $c^i = \pi_1(c)|_{\mathbb{Z}^2\times\{z+i\}}$, and $e^i = \gamma(c_i)$. By assumption, we have that $c^0_{(x,y)} \neq \square$ and hence $\gamma(e^0)_{(x,y)} = \blacksquare$. As shown before, for every $(x',y') \in \mathbb{Z}^2$ we have that if $e^0_{(x',y')} = \blacksquare$ then $e^1_{(x',y')} = \square$ and $e^2_{(x',y')} = \square$. From this property we conclude that $e^1 = e^2 = \square^{\mathbb{Z}^2}$. By definition of γ we obtain that $c^1 = c^2 = \square^{\mathbb{Z}^2}$ as required. $\square$

Definition 6.11. *Given $c \in X_{GW}$ a subset $W \subset \mathbb{Z}^3$ is called a* ***wave*** *if for every $w \in W$, $\pi_2(c)_w$ is* ◨ *and there exists a function $\varphi : \mathbb{Z}^2 \to \mathbb{Z}$ such that $W = \{(x,y,z) \in \mathbb{Z}^3 \mid z = \varphi(x,y)\}$. and for every $u, v \in \mathbb{Z}^2$*

$$|\varphi(u) - \varphi(v)| \leq ||u - v||_1$$

Furthermore, a wave W is a ***good wave*** *if*

$$\sup_{u,v\in\mathbb{Z}^2} |\varphi(u) - \varphi(v)| \leq 1.$$

and a ***flat wave*** *if φ is constant.*

Remark 6.12. Let $c \in X_{GW}$ and assume that W_1 and W_2 are two different waves. Using rule (C-Structure I) we deduce that

$$\inf_{w\in W_1, w'\in W_2} ||w - w'||_1 > 2$$

In particular, all the waves are disjoint.

Claim 6.13. *Let $c \in X_{GW}$, if there exists $w \in \mathbb{Z}^3$ such that $\pi_2(c)_w =$* ◨ *then there exists a unique wave W such that $w \in W$.*

Proof. Let $w = (x,y,z)$. Let $K \subset \mathbb{Z}^2$ be a maximal set such that there is a 1-Lipschitz function $\varphi : K \to \mathbb{Z}$ satisfying $\varphi(x,y) = z$ and $\pi_2(c)_{(x,y,\varphi(x,y))} =$ ◨. If $K \neq \mathbb{Z}^2$ pick $(i,j) \notin K$ at distance 1 from a point $(x_1,y_1) \in K$. As $\pi_2(c)_{(x_1,y_1,\varphi(x_1,y_1))} =$ ◨, rule (Σ-Structure II) implies the existence of a ◨ at exactly one of the positions $(i,j,\varphi(x_1,y_1)-1)$, $(i,j,\varphi(x_1,y_1))$ or $(i,j,\varphi(x_1,y_1)+1)$. Therefore one can extend φ to a bigger domain contradicting the maximality of K. The uniqueness is direct from the previous remark. $\square$

Claim 6.14. *Let $c \in X_{GW}$. Every wave is a good wave.*

Proof. Let W be a wave and $\varphi : \mathbb{Z}^2 \to \mathbb{Z}$ a function defining W. As φ is 1-Lipschitz its image is an interval in $\mathbb{Z}$. If $\varphi(\mathbb{Z}^2) = \{z\}$, $\varphi(\mathbb{Z}^2) = \{z, z+1\}$ or $\varphi(\mathbb{Z}^2) = \{z-1, z\}$ then the claim is satisfied. Otherwise, there are three contiguous values $\{\eta-1, \eta, \eta+1\}$ in the image of φ. Therefore there must be two adjacent positions u, v in $\mathbb{Z}^2$ which have $\varphi(u) = \eta - 1$ and $\varphi(v) = \eta$. Rule ($\Sigma \times C$-Structure III) implies that a wire tile appears in the $\eta - 1$ coset. On the other hand, the same argument applies to the pair $\eta, \eta+1$, showing that there is also a wire tile in the η-coset. Using Claim 6.10 we obtain a contradiction. $\square$

Claim 6.15. $\mathtt{A}_2(X_{\mathtt{GW}}, \sigma) \neq X^2_{\mathtt{GW}}$.

Proof. Clearly $(\square, \boxplus)^{\mathbb{Z}^3} \in X_{\mathtt{GW}}$. Let $c \in X_{\mathtt{GW}} \setminus \{(\square, \boxplus)^{\mathbb{Z}^3}\}$. We claim there is a coordinate $w \in \mathbb{Z}^3$ such that $\pi_2(c)_w =$ ❒. If not, as $c \neq (\square, \boxplus)^{\mathbb{Z}^3}$ there is $w' \in \mathbb{Z}^3$ such that $\pi_1(c)_{w'} \neq \square$ and thus by rule ($\Sigma \times C$-Structure I) there is also a ❒ in $\pi_2(c)$ appearing at some coordinate w. By Claim 6.13 there is a wave W for c which is furthermore a good wave by Claim 6.14. Let $c' \in X_{\mathtt{GW}}$ such that $(c, c') \in \mathtt{A}_1(X_{\mathtt{GW}}, \sigma)$ and let $(\tilde{c}, \tilde{c}')$ be an asymptotic pair such that $c|_{w+[-2,2]^3} = \tilde{c}|_{w+[-2,2]^3}$ and $c'|_{w+[-2,2]^3} = \tilde{c}'|_{w+[-2,2]^3}$. As $(\tilde{c}, \tilde{c}')$ are asymptotic they must coincide outside some finite region $F \Subset \mathbb{Z}^3$. Thus there must be a ❒ in $\pi_2(\tilde{c}')_{\tilde{w}}$ for some $\tilde{w} \in W \cap (\mathbb{Z} \setminus F)$. Again, there is a good wave $\widetilde{W}$ for $\tilde{c}'$ passing through $\tilde{w}$ and thus $\widetilde{W} \cap (w + [-2,2]^3) \neq \emptyset$. This shows that there is a ❒ appearing in $c'|_{w+[-2,2]^3}$.

We have so far shown that if $c \in X_{\mathtt{GW}} \setminus \{(\square, \boxplus)^{\mathbb{Z}^3}\}$ and $(c, c') \in \mathtt{A}_1(X_{\mathtt{GW}}, \sigma)$, then $c' \neq (\square, \boxplus)^{\mathbb{Z}^3}$. If $(c, (\square, \boxplus)^{\mathbb{Z}^3}) \in \mathtt{A}_2(X_{\mathtt{GW}}, \sigma)$, there would be a finite chain $c_0, c_1, \dots, c_n$ such that $c_0 = c$, $c_n = (\square, \boxplus)^{\mathbb{Z}^3}$ and $(c_i, c_{i+1}) \in \mathtt{A}_1(X_{\mathtt{GW}}, \sigma)$. By the previous argument every c_i in the chain must be $(\square, \boxplus)^{\mathbb{Z}^3}$ and thus $c = (\square, \boxplus)^{\mathbb{Z}^3}$ yielding a contradiction. This shows that $(c, (\square, \boxplus)^{\mathbb{Z}^3}) \notin \mathtt{A}_2(X_{\mathtt{GW}}, \sigma)$. □

Definition 6.16. *Let $c \in X_{\mathtt{GW}}$ and W be a good wave. Let φ be the function defining W. The* ***crest*** *of W is the set $W_c = \{(x, y, \varphi(x, y)) \in W \mid \varphi(x, y) = \max_{(i,j) \in \mathbb{Z}^2} \varphi(i, j)\}$.*

Let $c \in X_{\mathtt{GW}}$ and W a good wave which is not flat. From rules ($\Sigma \times C$-Structure II) and ($\Sigma \times C$-Structure III) there is a correspondence between crests and red wires. See Figure 11. On the other hand, note that flat waves correspond to planes in X_{struct} which either do not contain wires or only contain wires of a single color.

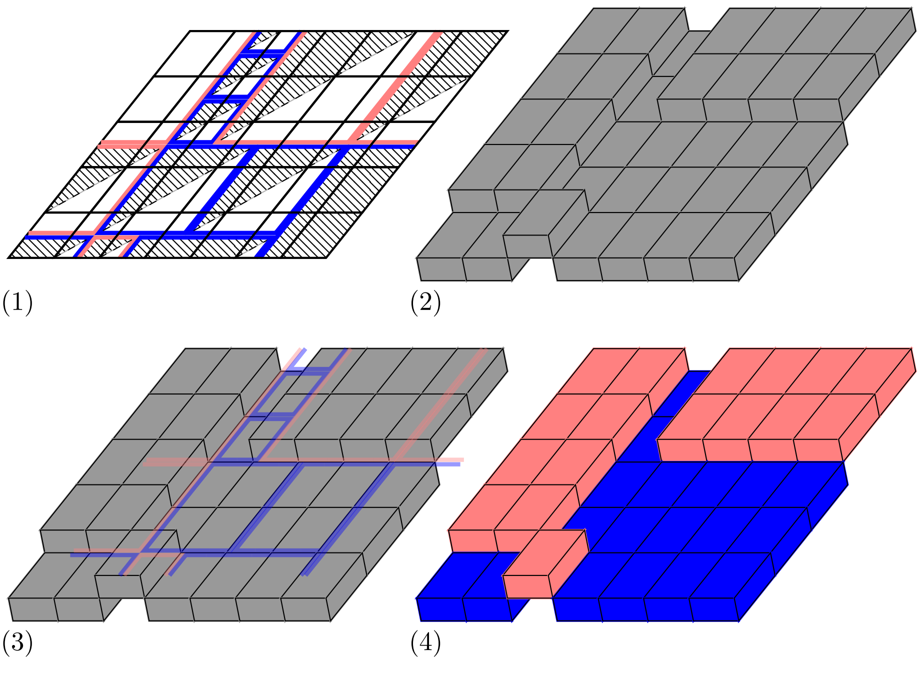


Figure 11: Different views of a configuration $c \in X_{\mathtt{GW}}$. In (1) we see a configuration in X_{struct}. (2) shows a good wave W. In (3) we see the superposition of both. In (4) we show the crest of the good wave in red to remark the correspondence.

Claim 6.17. *Let $c, c' \in X_{\mathtt{GW}}$ and $k \in \mathbb{Z}$. There is a configuration $\hat{c} \in X_{\mathtt{GW}}$ such that:*

1. *For every $z \in \mathbb{Z}$ such that $z < k - 2$ and $(x, y) \in \mathbb{Z}^2$, $\hat{c}_{(x,y,z)} = c_{(x,y,z)}$.*
2. *For every $z \in \mathbb{Z}$ such that $z > k + 2$ and $(x, y) \in \mathbb{Z}^2$, $\hat{c}_{(x,y,z)} = c'_{(x,y,z)}$.*

Proof. Let us begin by giving a sketch of how we construct $\hat{c}$. First, we use the coordinates $k - 2$ and $k + 2$ to complete any good waves which might appear partially in the coordinates $k - 3$ of c or $k + 3$ of c'. Second, we fill the remaining three undefined coordinates $k - 1, k, k + 1$ with the pair $(\square, \boxplus)$.

We shall now proceed formally. We begin by constructing a configuration $\tilde{c} \in X_{\mathtt{GW}}$ such that for $z < k-2$ $\tilde{c}_{(x,y,z)} = c_{(x,y,z)}$ and for $z > k-2$ $\tilde{c}_{(x,y,z)} = (\square,$⊞$)$. Consider the z-plane at $k-3$ and the configuration c. If there are no wire tiles nor ⧉s then $\tilde{c}_{(x,y,z)}$ can be defined as $c_{(x,y,z)}$ if $z < k-2$ and ($\square$, ⊞) otherwise. One can check directly that $\tilde{c}$ belongs to $X_{\mathtt{GW}}$ and satisfies the requirement. Otherwise, there is either (a) a coordinate $(x, y, k-3)$ where a non-white tile appears or (b) no wire tile appears in the z-plane at $k-3$ but a ⧉ appears in some coordinate of the form $(x, y, k-3)$.

In case (a), then by Claim 6.10 the z-planes at $k-2$ and $k-1$ consists uniquely of the white tile $\square$. Furthermore, the non-white tile forces a ⧉ to appear either at $(x, y, k-3)$ or $(x, y, k-2)$. Following the same argument as in Claim 6.14 we obtain that there is a plane of ⧉ completely contained in the coordinates $k-3$ and $k-2$, therefore by rule (C-Structure I) the z-plane at $k-1$ contains only ⊞. Therefore the configuration $\tilde{c}_{(x,y,z)}$ defined as $c_{(x,y,z)}$ if $z \leq k-2$ and ($\square$, ⊞) otherwise belongs to $X_{\mathtt{GW}}$ and coincides with c up to the z-plane at $k-1$.

In case (b) we have a ⧉ at a position $(x, y, k-3)$. As no wire tiles appear in the z-plane at $k-3$, we know there cannot be any ⧉ in the z-plane at $k-2$, therefore the plane of ⧉ is either completely contained in $z = k-3$, or part of it is in the z-plane at $k-4$. In either case, the configuration $\tilde{c}_{(x,y,z)}$ defined as $c_{(x,y,z)}$ if $z \leq k-3$ and ($\square$, ⊞) otherwise belongs to $X_{\mathtt{GW}}$ and coincides with c up to the z-plane at $k-3$.

We have so far a configuration $\tilde{c}$ such that for $z < k-2$, it is the case that $\tilde{c}_{(x,y,z)} = c_{(x,y,z)}$ and for $z > k-2$ we have $\tilde{c}_{(x,y,z)} = (\square,$⊞$)$. Similarly, one can construct a configuration $\tilde{c}'$ such that for $z > k+2$, $\tilde{c}'_{(x,y,z)} = c'_{(x,y,z)}$ and for $z < k+2$, $\tilde{c}'_{(x,y,z)} = (\square,$⊞$)$. We define $\hat{c}$ by:

$$\hat{c} = \begin{cases} \tilde{c}_{(x,y,z)} & \text{if } z \leq k \\ \tilde{c}'_{(x,y,z)} & \text{if } z > k \end{cases}$$

One can verify that $\hat{c} \in X_{\mathtt{GW}}$ and that it satisfies the requirements of the claim. $\square$

Claim 6.18. $\mathtt{A}_3(X_{\mathtt{GW}}, \sigma) = X^2_{\mathtt{GW}}$

Proof. The proof proceeds in four steps: in the first three steps we show that if two configurations in $X_{\mathtt{GW}}$ have the same finite amount of good waves, then they belong to $\mathtt{A}_2(X_{\mathtt{GW}}, \sigma)$. Then, we show that the set of all such pairs is dense in $X^2_{\mathtt{GW}}$.

Step 1 Two configurations having exactly one flat wave and no non-white tiles.

More precisely, let $k \in \mathbb{Z}$ and $c^k \in X_{\mathtt{GW}}$ be the configuration:

$$c^k_{(x,y,z)} = \begin{cases} (\square, ⧉) & \text{if } z = k \\ (\square, ⊞) & \text{otherwise.} \end{cases}$$

We will show that for any pair k, k', then $(c^k, c^{k'}) \in \mathtt{A}_2(X_{\mathtt{GW}}, \sigma)$. As this relation is transitive and shift-invariant, it suffices to show that $(c^0, c^1) \in \mathtt{A}_1(X_{\mathtt{GW}}, \sigma)$.

Let $n \in \mathbb{N}$, $F_n = [-n, n]^3$ and consider the pattern $p^k_n = c^k|_{F_n}$. We can construct asymptotic configurations $\tilde{c}^0$ and $\tilde{c}^1$ such that $\tilde{c}^0|_{F_n} = p^0_n$ and $\tilde{c}^1|_{F_n} = p^1_n$. The construction which follows is illustrated in Figure 12.

Indeed, we define $\tilde{c}^0$ as follows: On the first layer the configuration contains only the $\square$ tile in every z-coset but on $z = 0$. On $z = 0$ it contains a blue square with no wire tiles on the support $[-3n+1, n+1]^2$ and the blue cross wire tile everywhere else. In the second layer it is equal to c^0.

Define $\tilde{c}^1$ as follows: On the first layer the configuration contains only the $\square$ tile in every z-coset but on $z = 0$. On $z = 0$ it contains a red square with no wire tiles on the support $S = [-3n+1, n+1]^2$ and the blue cross wire tile everywhere else. In the second layer it contains ⧉ only on the $z = 0$ and $z = 1$ cosets following the first layer on $z = 0$. That is, there is a ⧉ on $(x, y, 0)$ if and only if $(x, y) \in \mathbb{Z}^2 \setminus (S + (1, 1))$ and on $(x, y, 1)$ if and only if $(x, y) \in (S + (1, 1))$.

Clearly, we have that $\tilde{c}^0|_{\mathbb{Z}^3 \setminus F_{4n}} = \tilde{c}^1|_{\mathbb{Z}^3 \setminus F_{4n}}$ and thus they are asymptotic. Also $\tilde{c}^0|_{F_n = p^0_n}$ and $\tilde{c}^1|_{F_n = p^1_n}$. As n is arbitrary, we conclude $(c^0, c^1) \in \mathtt{A}_1(X_{\mathtt{GW}}, \sigma)$. Therefore $(c^0, c^k) \in \mathtt{A}_2(X_{\mathtt{GW}}, \sigma)$.

Step 2 Two configurations having exactly one good wave.

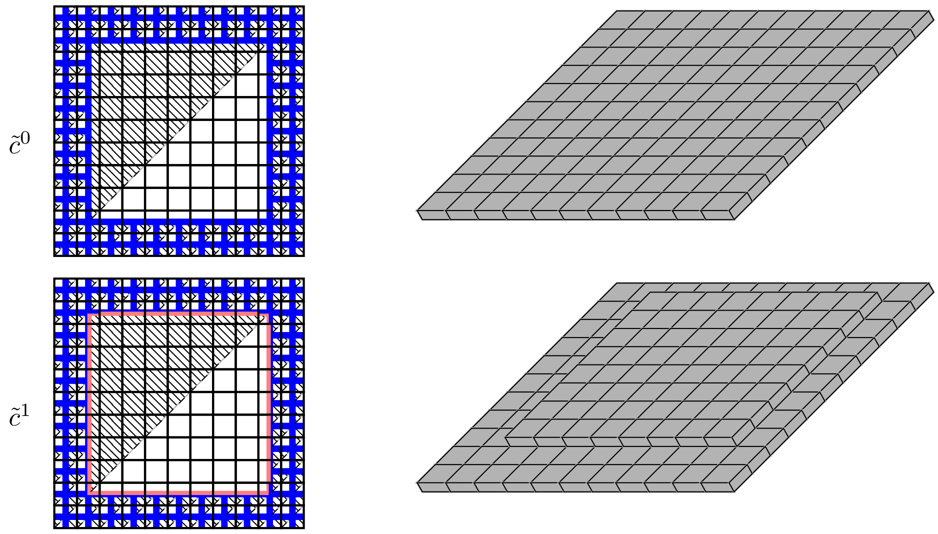


Figure 12: The asymptotic configurations $\tilde{c}^0$ (above) and $\tilde{c}^1$ (below). The only difference on the first coordinate is the color of the boundary of the inner square. Consequently, the wave at $z = 0$ is flat in $\tilde{c}^0$ but has a crest on $\tilde{c}^1$.

Now consider a configuration c which has only one good wave. If there are no non-white tiles in c the wave must be flat and thus $c = c^k$ for some $k \in \mathbb{Z}$ as defined in step 1. Otherwise, there is tile which is not white at some position (x, y, k). By rule ($\Sigma \times C$-Structure I) we know that there must be a ◨ appearing either in (x, y, k) or $(x, y, k+1)$. By Claim 6.10 we know that the non-white tiles are then constrained to have the same third coordinate k.

Let n be large enough such that $|k| < n$. Let $F_n = [-n, n]^3$, H the z-plane at k and consider $p = \pi_1(c)|_{F_n \cap H}$. There are two cases to consider: (a) there is a wire tile in p, (b) there are no wire tiles in p. By Claim 6.9 we can construct a configuration $\hat{x} \in X_{\text{struct}}$ such that $\hat{x}|_{[-n,n]^2} = p$ and is asymptotic to ⊞$^\infty$. In case (a) the colors of every wire in $\hat{x}$ are determined already, whereas in (b) we can choose the color of the wires surrounding the support $[-n, n]^2$. If a ◨ appears at $(0, 0, k)$ we set it blue, whereas if it appears at $(0, 0, k+1)$ we set it red. We can define $\hat{c} \in X_{\texttt{GW}}$ by setting in the first layer the $z = k$ coset to be $\hat{x}$ and □ everywhere else. $\pi_2(\hat{c})$ contains only ◨s in the z-planes at k and $k+1$ consistently with the colors of the wires in $\hat{x}$. One can check that with these choices $\hat{c}|_{F_n} = c|_{F_n}$. On the other hand, the configuration $\tilde{c}^k$ constructed in step 1 is asymptotic to $\hat{c}$. As n is arbitrary we obtain that $(c, c^k) \in \mathtt{A}_1(X_{\texttt{GW}}, \sigma)$. Joining this with step 1 we obtain that every pair of configurations (c, c') which contain a unique good wave belong to $\mathtt{A}_2(X_{\texttt{GW}}, \sigma)$.

Step 3 Two configurations with exactly m good waves.

Let c be a configuration with exactly m good waves. The idea is to think of c as a superposition of m disjoint good waves and modify one wave at a time to turn c into a configuration with m flat waves appearing at fixed positions. To achieve this we use step 2 to get rid of non-white tiles and turn the waves flat and step 1 to move flat waves to a fixed set of positions. This must be done carefully to avoid creating forbidden patterns. For instance, if we consider the configuration containing a flat wave at $z = 0$ with no non-white tiles and a flat wave at $z = 3$ with only red cross tiles on $z = 2$, then step 2 does nothing to the flat wave at $z = 0$, while it may turn the flat wave at $z = 3$ into a new flat wave on $z = 2$ thus creating two ◨ at distance 2 apart and violating rule (C-Structure I). This possible conflict is solved by operating from the lowest coordinate towards the largest one and leaving enough space between good waves at every step.

Let us proceed formally. Let $k_1 < \cdots < k_m$ be the set of $k_i \in \mathbb{Z}$ such that $\pi_2(c)_{(0,0,k_i)} =$ ◨, let $N > 3m + \max(|k_1|, |k_m|)$ and let $\hat{c}$ be the configuration which has exactly m flat waves at positions $-N, -N+3, \ldots, -N+3(m-1)$ and no non-white tiles. We claim that it is enough to show that $(c, \hat{c}) \in \mathtt{A}_2(X_{\texttt{GW}}, \sigma)$. Indeed, if c_1 and c_2 are two configurations which exactly m waves, we can choose N large enough to satisfy the above requirement for both of them and thus $(c_1, c_2) \in \mathtt{A}_2(X_{\texttt{GW}}, \sigma)$.

In order to show that $(c, \hat{c}) \in \mathtt{A}_2(X_{\mathtt{GW}}, \sigma)$ we construct a finite sequence of configurations $\{c_i\}$ which begins with c, ends with $\hat{c}$ and so that c_{i+1} can be obtained from c_i by applying one of the previous two steps, that is, such that $(c_i, c_{i+1}) \in \mathtt{A}_2(X_{\mathtt{GW}}, \sigma)$. More precisely, each c_{i+1} is obtained from c_i through a step in the following procedure:

For $j = 1, \dots, m$:

1. Use step 2 to turn the good wave at k_j into a flat wave on the least possible z-coordinate and erase its associated non-white tiles.

2. Iterate step 1 to move the flat wave obtained in the previous step to the coordinate $-N+3(j-1)$.

Clearly the configuration obtained at the end of this procedure is $\hat{c}$. Let us argue that at every step the configuration obtained from the previous one contains no forbidden patterns.

If $j = 1$, then step 2 is used on the good wave at k_1. On the one hand the non-white tiles are eliminated, and the s can only decrease their z-coordinate, thus no forbidden patterns can be created. The iterative application of step 1 only increases the distance between the waves and thus does not create any forbidden patterns.

If $j > 1$, we are in a configuration where the first $(j-1)$ waves are flat, have no associated non-white tiles, and are located at $-N, -N+3, \dots, -N+3(j-2)$. The application of step 2 on the j-th good wave again does not decrease the distance towards the $(j+1)$-th wave and by definition the distance with the $(j-1)$-th wave is at least 3. This also holds for every application of step 1. Therefore no forbidden patterns appear.

Step 4 An arbitrary pair of configurations $(c_1, c_2) \in X_{\mathtt{GW}}^2$.

We have shown so far that every pair of configurations having the same amount of good waves belongs to $\mathtt{A}_2(X_{\mathtt{GW}}, \sigma)$. It suffices to show that the pairs of configurations with the same amount of good waves are dense in $X_{\mathtt{GW}}^2$.

Indeed, let $n \in \mathbb{N}$ and use Claim 6.17 with c_1 and $(\square,)^{\mathbb{Z}^3}$ and $k = n$ to construct a configuration in $X_{\mathtt{GW}}$ which is equal to c_1 for all $z < n-2$ and has $(\square,)$ at any coordinate with $z > n+2$. Then use again Claim 6.17 with $(\square,)^{\mathbb{Z}^3}$ and this configuration to obtain a configuration $\hat{c}_1^n \in X_{\mathtt{GW}}$ which is equal to c_1 for $|z| < n-2$ and has $(\square,)$ at any coordinate with $|z| > n+2$. Do the same procedure to construct $\hat{c}_2^n \in X_{\mathtt{GW}}$ with the same properties but changing c_1 by c_2.

Now, both $\hat{c}_1^n$ and $\hat{c}_2^n$ have a finite amount of good waves. Up to renaming, assume that $\hat{c}_1^n$ has at least as many good waves as $\hat{c}_2^n$ and define $\tilde{c}_1^n = \hat{c}_1^n$. If $\hat{c}_1^n$ and $\hat{c}_2^n$ have the same amount of good waves, define $\tilde{c}_2^n = \hat{c}_2^n$. Otherwise, let r be the difference in the amount of good waves and define $\tilde{c}_2^n$ at (x, y, z) as $\hat{c}_2^n$ if $z \leq n+2$. If $z > n+2$ we put the white tile everywhere and a if and only if $z = n+2+3j$ for $j = 1, \dots, r$. That is, we add artificially r flat waves to the right of $\hat{c}_2^n$. Clearly $\tilde{c}_2^n \in X_{\mathtt{GW}}$.

We have that $\tilde{c}_1^n$ and $\tilde{c}_2^n$ have the same amount of good waves, hence by step 3 $(\tilde{c}_1^n, \tilde{c}_2^n) \in \mathtt{A}_2(X_{\mathtt{GW}}, \sigma)$. Moreover, by definition we have that $\tilde{c}_1^n|_{F_{n-3}} = c_1|_{F_{n-3}}$ and $\tilde{c}_2^n|_{F_{n-3}} = c_2|_{F_{n-3}}$. Therefore the sequence of pairs $\{(\tilde{c}_1^n, \tilde{c}_2^n)\}_{n \in \mathbb{N}}$ converges to (c_1, c_2). This shows that $(c_1, c_2) \in \mathtt{A}_3(X_{\mathtt{GW}}, \sigma)$. □

Proof of Theorem 6.8. Consider the Good Wave Shift $X_{\mathtt{GW}}$. Claim 6.15 shows $\mathtt{A}_2(X_{\mathtt{GW}}, \sigma) \neq X_{\mathtt{GW}}^2$ and Claim 6.18 that $\mathtt{A}_3(X_{\mathtt{GW}}, \sigma) = X_{\mathtt{GW}}^2$. Thus $X_{\mathtt{GW}}$ is in the asymptotic class 3. □

In order to answer Pavlov's question, we need to show that $X_{\mathtt{GW}}$ has topological completely positive entropy and is not bounded chain exchangeable. This is the purpose of the following two claims.

Claim 6.19. *There is a shift-invariant fully supported measure in $X_{\mathtt{GW}}$.*

Proof. A configuration c is strongly periodic if its orbit is finite. We claim that the strongly periodic configurations are dense in $X_{\mathtt{GW}}$. Indeed, let $p \in L(X_{\mathtt{GW}})$ be an admissible pattern with support $F_n = [-n, n]^3$. There is always a way to extend p to a pattern $q \in L(X_{\mathtt{GW}})$ with support F_{4n} such that the boundary of the intersection with every z-plane consists uniquely of either:

$$(,), (\square,) \text{ or } (\square,)$$

and such that the z-planes at $-4n, -4n+1, 4n-1$ and $4n$ consist uniquely on $(\square,)$. Indeed, let us give a sketch of the proof. If the intersection with a z-plane contains a non-white tile we may use Claim 6.9 to extend it so that the boundary consists uniquely of the pair $(,)$. On the remaining coordinates there can only be flat waves or no waves at all and thus it is easy to extend them so that the boundary is composed entirely either of $(\square, \blacksquare)$ or $(\square, \boxplus)$. We use the remaining z-planes to complete any wave appearing partially in p and fill the rest with the pair $(\square, \boxplus)$. This pattern can be extended to a strongly periodic configuration $c \in X_{\texttt{GW}}$ by setting $c_{(i,j,k)} := q(((i,j,k) \mod 8n+1)-(4n,4n,4n))$. This shows that strongly periodic configurations are dense.

Constructing a fully supported measure from a dense set of strongly periodic configurations is standard, we give a proof for completeness. Consider a dense sequence of strongly periodic configurations $\{x_n\}_{n\in\mathbb{N}}$. For each configuration x_n consider the uniform measure μ_{x_n} supported on the finite orbit of x_n. We claim that $\mu = \sum_{n>0} 2^{-n}\mu_{x_n}$ is a shift-invariant fully supported measure. Let $x \in X_{\texttt{GW}}$ and pick a neighborhood $U \in \mathcal{N}_x$. As $\{x_n\}_{n\in\mathbb{N}}$ is dense, there exists N such that $x_N \in U$. Therefore

$$\mu(U) \geq \frac{2^{-N}}{|\operatorname{Orb}(x_N)|} > 0.$$

And thus $x \in \operatorname{supp}(\mu)$. On the other hand, μ is obviously shift-invariant. □

Claim 6.20. *$X_{\texttt{GW}}$ is topologically weakly mixing.*

Proof. Let p, q be patterns in $L(X_{\texttt{GW}})$ with support F and p', q' patterns in $L(X_{\texttt{GW}})$ with support F'. Let m be the largest z-coordinate of an element of F and $k = m+3$. Choose a vector $u \in \mathbb{Z}^3$ such that the third coordinate of any element in $u + F'$ is strictly larger than $k+2$. Choose $c \in [p], \tilde{c} \in [q], c' \in \sigma^{-u}([p']), \tilde{c}' \in \sigma^{-u}([q'])$. By Claim 6.17 there is a configuration $\hat{c}$ which coincides with c for $z < k-2$ and with $\tilde{c}$ for $z > k+2$. Similarly, there is $\hat{c}'$ which coincides with c' for $z < k-2$ and with $\tilde{c}'$ for $z > k+2$. We clearly have that $(\tilde{c}, \tilde{c}') \in ([p] \times [q]) \cap \sigma^{-u}([p'] \times [q'])$. Thus showing that $X_{\texttt{GW}}^2$ is transitive. □

Theorem 6.21. *There is a topologically weakly mixing $\mathbb{Z}^3$-SFT in the CPE class 3.*

Proof. Consider the Good Wave Shift $X_{\texttt{GW}}$. Claim 6.20 says that $X_{\texttt{GW}}$ is topologically weakly mixing and Theorem 6.8 says it is in the asymptotic class 3. Furthermore Claim 6.19 says it admits a fully supported measure, therefore by Corollary 4.7 we conclude it is in the CPE class 3. □

Theorem 6.22. *There is a topologically weakly mixing $\mathbb{Z}^3$-SFT with topological CPE which does not have BCE.*

Proof of Theorem 6.22. Consider the Good Wave Shift $X_{\texttt{GW}}$. By Theorem 6.21 $X_{\texttt{GW}}$ is topologically weakly mixing and in the CPE class 3. By Theorem 2.4 it must have topological CPE. On the other hand, Theorem 6.8 says $X_{\texttt{GW}}$ is in the asymptotic class 3. Proposition 6.18 implies that it cannot have BCE. □

As a matter of fact this yields the first known example of a transitive $\mathbb{Z}^d$-SFT with topological CPE but which does not have UPE.

Question 6.23. *Is there a $\mathbb{Z}^2$-SFT in the CPE class 3?*

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