A generalization of the simulation theorem for semidirect products

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Abstract

We generalize a result of Hochman in two simultaneous directions: Instead of realizing an effectively closed $\mathbb{Z}^d$ action as a factor of a subaction of a $\mathbb{Z}^{d+2}$-SFT we realize an action of a finitely generated group analogously in any semidirect product of the group with $\mathbb{Z}^2$. Let H be a finitely generated group and $G = \mathbb{Z}^2 \rtimes H$ a semidirect product. We show that for any effectively closed H-dynamical system (Y, f) where Y is a Cantor set, there exists a G-subshift of finite type (X, σ) such that the H-subaction of (X, σ) is an extension of (Y, f). In the case where f is an expansive action, a subshift conjugated to (Y, f) can be obtained as the H-projective subdynamics of a G-sofic subshift. As a corollary, we obtain that G admits a non-empty strongly aperiodic subshift of finite type whenever the word problem of H is decidable.

1 Introduction

A dynamical system is a tuple (X, T) where X is a set and $T : X \to X$ is a map which describes the evolution of points of X in time. In the case where T is bijective one can describe T as a $\mathbb{Z}$-action by associating $(n, x) \to T^n(x)$. This can be generalized to a set of bijective maps $T_1, \ldots, T_n$ which satisfy some set of relations R –for instance, the relation $T_1 \circ T_2 = T_2 \circ T_1$ which indicates T_1 and T_2 commute–. These actions and their relations can be expressed by the group action $\mathcal{T} : G \times X \to X$ where $G \cong \langle T_1, \ldots, T_n \mid R \rangle$ and $\mathcal{T}(T_{i_1} \circ \cdots \circ T_{i_k}, x) = T_{i_1} \circ \cdots \circ T_{i_k}(x)$.

More than often dynamical systems arising from group actions are difficult to study, and a fruitful technique is to look at their subactions, that is, the restriction of the group action to a particular subgroup. For instance, see the study of expansive subdynamics of $\mathbb{Z}^d$ actions [6, 10]. It is thus appealing to ask the following question: What systems can be obtained as subactions of a

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class of dynamical systems? An interesting class is the one of subshifts of finite type (SFT), that is, the sets of colorings of a group along with the shift action which are defined by a finite number of forbidden patterns.

For the class of $\mathbb{Z}^d$-SFTs there is still no characterization of which dynamical systems can arise as their subactions, nevertheless, it has been proven by Hochman [12] that every $\mathbb{Z}^d$-action over a Cantor set $T : \mathbb{Z}^d \times X \to X$ which is effectively closed – meaning that it can be described with a Turing machine– admits an almost trivial isometric extension which can be realized as the subaction of a $\mathbb{Z}^{d+2}$-SFT. This result has subsequently been improved for the expansive case independently in [3] and [9] showing that every effectively closed subshift can be obtained as the projective subdynamics of a sofic $\mathbb{Z}^2$-subshift. These kind of results yield powerful techniques to prove properties about the original systems. An example is the characterization of the set of entropies of $\mathbb{Z}^2$-SFTs [13] as the set of right recursively enumerable numbers.

In this article we extend Hochman's result to the case of group actions for groups which are of the form $G = \mathbb{Z}^2 \rtimes_\varphi H$ for some finitely generated group H and a homomorphism $\varphi : H \to \mathrm{Aut}(\mathbb{Z}^2)$. More specifically we prove the following result.

Theorem 3.1. *For every H-effectively closed dynamical system (X, f) there exists a $(\mathbb{Z}^2 \rtimes H)$-SFT whose H-subaction is an extension of (X, f).*

We remark the strong gap which occurs when passing from $\mathbb{Z}$-SFTs to the multidimensional case. For instance, $\mathbb{Z}$-SFTs contain periodic points, have regular languages and the possible set of entropies they can have is reduced to logarithms of Perron numbers [17]. In the other hand multidimensional SFTs can be strongly aperiodic [5, 20, 16, 15], can be composed uniquely of non-computable points [11, 19] and their entropies are not even computable [13]. Most of these differences can be put into evidence with simulation theorems by the fact that multidimensional SFTs can be projected onto effectively closed subshifts in one dimension. Our Theorem 3.1 allows analogously to extend properties of effectively closed subshifts in general groups H and show that they also appear in SFTs when the group is replaced by $\mathbb{Z}^2 \rtimes H$. This is a powerful tool to construct SFTs with some desired property in an arbitrary finitely generated group as long as said property is present in some effectively closed subshift.

Readers who are not familiar with computability or the embedding of Turing machine computations in subshifts of finite type will be reassured by the fact that in the proof all of those aspects are hidden in black boxes. Namely, we use the result of [3, 9] that every effectively closed $\mathbb{Z}$-subshift is the projective subdynamics of a sofic $\mathbb{Z}^2$-subshift whose vertical shift action is trivial. We also make use of a theorem of Mozes [18] which states that subshifts arising from two-dimensional substitutions are sofic.

In the case when H is a recursively presented group, Theorem 3.1 can be presented in a purely symbolic dynamics fashion for expansive actions, namely we show:

Theorem 4.3. *Let X be an effectively closed H-subshift. Then there exists a sofic $(\mathbb{Z}^2 \rtimes H)$-subshift Y such that its H-projective subdynamics $\pi_H(Y)$ is X.*

It is known that every $\mathbb{Z}$-SFT contains a periodic configuration [17]. However, it was shown by Berger [5] that there are $\mathbb{Z}^2$-SFTs which are strongly aperiodic, that is, such that the shift acts freely on the set of configurations. This result has been proven several times with different techniques [20, 16, 15] giving a variety of constructions. However, it remains an open question which is the class of groups which admit strongly aperiodic SFTs. Amongst the class of groups that do admit strongly aperiodic SFTs are: $\mathbb{Z}^d$ for $d > 1$, hyperbolic surface groups [8], Osin and Ivanov monster groups [14], and the direct product $G \times \mathbb{Z}$ for a particular class of groups G which includes Thompson's T group and $\mathrm{PSL}(\mathbb{Z}, 2)$ [14]. It is also known that no group with two or more ends can contain strongly aperiodic SFTs [7] and that recursively presented groups which admit strongly aperiodic SFTs must have decidable word problem [14].

As an application of Theorem 3.1 we present a new class of groups which admit strongly aperiodic SFTs, that is:

Theorem 4.4. *Every semidirect product $\mathbb{Z}^2 \rtimes H$ where H is finitely generated and has decidable word problem admits a non-empty strongly aperiodic SFT.*

Amongst this new class of groups which admit strongly aperiodic SFTs, we remark the discrete Heisenberg group which admits a presentation $\mathcal{H} \cong \mathbb{Z}^2 \rtimes \mathbb{Z}$. A construction by Ugarcovici, Sahin and Schraudner showing that $\mathcal{H}$ admits strongly aperiodic SFTs was already presented in a workshop [21] in 2014. Our results provide a new proof of this result along with a positive answer to their question asking if similar constructions can be realized in the powers of the Heisenberg group, the Flip group and the Sol group.

2 Preliminaries

Consider a group G and a compact topological space $(X, \mathcal{T})$. The tuple (X, f) where $f : G \times X \to X$ is a left G action by homeomorphisms is called a G-dynamical system. Let (X, f), (X', f') be two G-dynamical systems. We say $\phi : X \to X'$ is a *morphism* if it is continuous and $\phi \circ f_g = f'_g \circ \phi$ for all $g \in G$. A surjective morphism $\phi : X \twoheadrightarrow X'$ is a *factor* and we say that (X', f') is a *factor* of (X, f) and that (X, f) is an *extension* of (X', f'). When ϕ is a bijection and its inverse is continuous we say it is a *conjugacy* and that (X, f) is *conjugated* to (X', f').

In what follows, we consider the space X to be a Cantor set equipped with the product topology and a finitely generated group acting over X. Without loss of generality, we consider X to be a closed subset of $\{0,1\}^{\mathbb{N}}$. Let G be a group generated by a finite set S. A *G-effectively closed dynamical system* is a G-dynamical system (X, f) where:

1. $X \subset \{0,1\}^{\mathbb{N}}$ is a closed effective subset: $X = \{0,1\}^{\mathbb{N}} \setminus \bigcup_{i \in I} [w_i]$ where $\{w_i\}_{i \in I} \subset \{0,1\}^*$ is a recursively enumerable language. That means that X is the complement of a union of cylinders which can be enumerated by a Turing machine.

2. f is an effectively closed action: there exists a Turing machine which on entry $s \in S$ and $w \in \{0,1\}^*$ enumerates a sequence of words $(w_j)_{j \in J}$ such that $f_s^{-1}([w]) = \{0,1\}^{\mathbb{N}} \setminus \bigcup_{j \in J}[w_j]$.

The idea behind the definition is the following: There is a Turing machine T which given a word $g \in S^*$ representing an element of G and n coordinates of $x \in X \subset \{0,1\}^{\mathbb{N}}$ returns an approximation of the preimage of x by f_g.

Let $\mathcal{A}$ be a finite alphabet and G a finitely generated group. The set $\mathcal{A}^G = \{x : G \to \mathcal{A}\}$ equipped with the left group action $\sigma : G \times \mathcal{A}^G \to \mathcal{A}^G$ given by: $(\sigma_h(x))_g = x_{h^{-1}g}$ is the *G-full shift*. The elements $a \in \mathcal{A}$ and $x \in \mathcal{A}^G$ are called *symbols* and *configurations* respectively. We endow $\mathcal{A}^G$ with the product topology, therefore obtaining a compact metric space. The topology is generated by the metric $d(x,y) = 2^{-\inf\{|g| \mid g \in G \colon x_g \neq y_g\}}$ where $|g|$ is the length of the smallest expression of g as the product of some fixed set of generators. This topology is also generated by a clopen basis given by the *cylinders* $[a]_g = \{x \in \mathcal{A}^G | x_g = a \in \mathcal{A}\}$. A *support* is a finite subset $F \subset G$. Given a support F, a *pattern with support* F is an element P of $\mathcal{A}^F$, i.e. a finite configuration and we write *supp*$(P) = F$. We also denote the cylinder generated by P centered in g as $[P]_g = \bigcap_{h \in F}[P_h]_{gh}$. If $x \in [P]_g$ for some $g \in G$ we write $P \sqsubset x$.

A subset X of $\mathcal{A}^G$ is a *G-subshift* if it is σ-invariant – $\sigma(G, X) \subset X$ – and closed for the cylinder topology. Equivalently, X is a G-subshift if and only if there exists a set of forbidden patterns $\mathcal{F}$ that defines it.

$$X = X_{\mathcal{F}} := \mathcal{A}^G \setminus \bigcup_{P \in \mathcal{F}, g \in G} [P]_g.$$

That is, a G-subshift is a subset of $\mathcal{A}^G$ which can be written as the complement of the orbit of a union of cylinders

If the context is clear enough, we will drop the group G from the notation and simply refer to a subshift. A subshift $X \subseteq \mathcal{A}^G$ is *of finite type* – SFT for short – if there exists a finite set of forbidden patterns $\mathcal{F}$ such that $X = X_{\mathcal{F}}$. A subshift $X \subseteq \mathcal{A}^G$ is *sofic* if there exists a subshift of finite type $Y \subset \mathcal{A}'^G$ and a factor $\phi : Y \twoheadrightarrow X$. A subshift is effectively closed if there exists a recursively enumerable coding of a set of forbidden patterns $\mathcal{F}$ such that $X = X_{\mathcal{F}}$. More details can be found in [2] or in Section 4.

Any G-dynamical system over a Cantor set can be seen as a subshift over an infinite alphabet: Indeed, (X, f) can be seen as $Y \subset (\{0,1\}^{\mathbb{N}})^G$ equipped with the shift action such that $x \in Y$ if and only if $\forall g \in G\ x_g = f_g(x_{1_G})$. In this setting, effectively closed G-dynamical systems correspond to effectively closed subshifts in this infinite alphabet.

Let $H \leq G$ be a subgroup and (X, f) a G-dynamical system. The *H-subaction* of (X, f) is (X, f^H) where $f^H : H \times X \to X$ is the restriction of f to H, that is $\forall h \in H, (f^H)_h(x) = f_h(x)$. In the case of a subshift $X \subset \mathcal{A}^G$ there is also the different notion of projective subdynamics. The *H-projective subdynamics* of X is the set $\pi_H(X) = \{y \in \mathcal{A}^H \mid \exists x \in X, \forall h \in H, y_h = x_h\}$. It is important to remark that subactions don't preserve expansivity, so in particular a subaction of a subshift is not necessarily a subshift. Nevertheless, the projective subdynamics of a subshift $\pi_H(X)$ is always an H-subshift.

Throughout this article we make use of the following notation. If $x \in \mathcal{A}^G$ is a configuration such that $\forall g \in F \subset G$ then $x_g = a \in A$ we just write $x|_F \equiv a$.

3 Simulation Theorem

The purpose of this section is to prove our main result.

Theorem 3.1. *Let H be finitely generated group and $G = \mathbb{Z}^2 \rtimes H$. For every H-effectively closed dynamical system (X, f) there exists a G-SFT whose H-subaction is an extension of (X, f).*

We begin by introducing some useful constructions. The general schema of the proof is the following: First, we construct for each non-zero vector $v \in (\mathbb{Z}/3\mathbb{Z})^2$ a substitution $\mathtt{s}_v$. Each configuration on the subshift generated by $\mathtt{s}_v$ encodes countably many copies of $\mathbb{Z}^2$ as lattices. These lattices are situated in a way such that any automorphism $\varphi \in \mathrm{Aut}(\mathbb{Z}^2)$ acting over the space of configurations by permuting the coordinates has as an image the subshift generated by $\mathtt{s}_{\tilde{\varphi}(v)}$, where $\tilde{\varphi} \in \mathrm{Aut}((\mathbb{Z}/3\mathbb{Z})^2)$ is the automorphism of $(\mathbb{Z}/3\mathbb{Z})^2$ obtained by reducing each entry of the matrix representation of φ modulo 3. The purpose of the lattices is to encode a finite amount of information, namely, each lattice will be later on paired to a specific coordinate of a configuration in $\{0,1\}^{\mathbb{N}}$ and will transmit this information when moving in G by elements of H.

The second ingredient of this proof is a joint encoding of the elements of X and the H-dynamical system f in an effective Toeplitz $\mathbb{Z}$-subshift. We do so in a way that the horizontal and vertical projections of the n-th order lattice of the previous construction always match with the n-th coordinate of $x \in X \subset \{0,1\}^{\mathbb{N}}$. For technical reasons of matching all the possible projections, we parametrize these Toeplitz subshifts with a natural number $q \in \{1, 2\}$.

Afterwards, we extend the Toeplitz subshift to a $\mathbb{Z}^2$-subshift by repeating rows (or columns). Using a known simulation theorem we obtain that this object is a sofic $\mathbb{Z}^2$-subshift from which we extract an SFT extension. We then proceed to couple this structure with the substitution subshifts described above in such a way that the symbols encoded by the Toeplitz layers match with the lattices of the substitution.

In the next step, we extend this construction to a G-SFT by adding local rules that ensure that if the $(\mathbb{Z}^2, 0)$-coset of a configuration y in said subshift codes $x \in X$ then for any $h \in H$ the $(\mathbb{Z}^2, 0)$-coset of $\sigma_h(y)$ codes $f_h(x)$. This set of rules is described as a finite amount of forbidden patterns.

Finally, we define the factor code, and show that it satisfies the required properties.

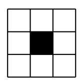
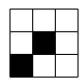
3.1 A set of $\mathbb{Z}^2$-substitutions which are permuted by actions of $\mathbf{Aut}(\mathbb{Z}^2)$.

Let $p \geq 3$ be an integer. We define a substitution over a two symbol alphabet which generates a sofic $\mathbb{Z}^2$-subshift encoding translations of $p^{m+1}\mathbb{Z}^2$ for $m \in \mathbb{N}$. In the proof of the simulation theorem we will only use the case where $p = 3$, but we prefer to proceed here with more generality.

To make notations shorter, we write $\vec{0} = (0,0) \in \mathbb{Z}^2$ throughout the whole proof. Let $v \in (\mathbb{Z}/p\mathbb{Z})^2 \setminus \{\vec{0}\}$ and $\mathcal{A} = \{\square, \blacksquare\}$. The $\mathbb{Z}^2$-substitution $\mathtt{s}_v : \mathcal{A} \to \mathcal{A}^{\{0,\dots,p-1\}^2}$ is defined by:

$$(\mathtt{s}_v(\square))_u = \begin{cases} \blacksquare & \text{if } u = v \\ \square & \text{otherwise.} \end{cases} \qquad (\mathtt{s}_v(\blacksquare))_u = \begin{cases} \blacksquare & \text{if } u \in \{\vec{0}, v\} \\ \square & \text{otherwise.} \end{cases}$$

As an example, if $p = 3$ and $v = (1,1)$ we get the following:

$\mathtt{s}_v(\square) =$  $\qquad \mathtt{s}_v(\blacksquare) =$ 

In this example the patterns $\mathtt{s}_v^3(\blacksquare)$ and $\mathtt{s}_v^4(\blacksquare)$ can be seen in Figure 1.

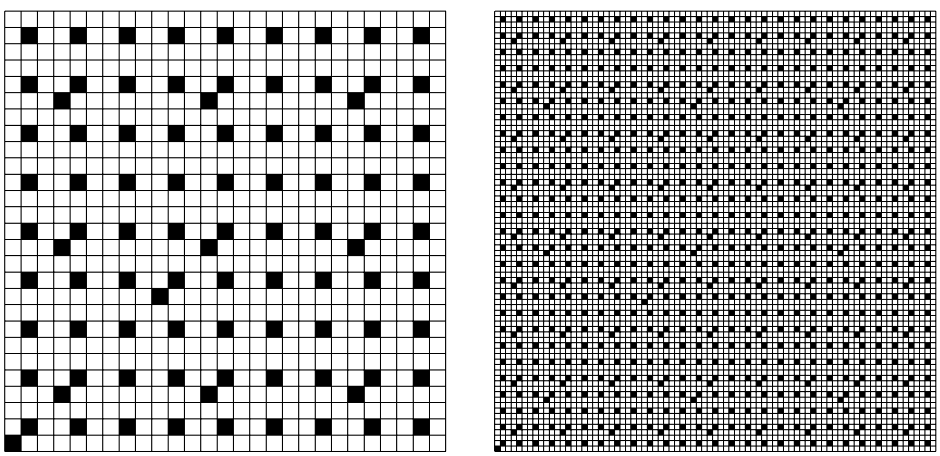


Figure 1: The patterns or order 3 and 4 of $\mathtt{s}_v$ for $v = (1,1)$.

To a substitution $\mathtt{s}_v$ we associate the subshift $\mathtt{Sub}_v$ defined as the set of $\mathbb{Z}^2$-configurations such that every subpattern appears in some iteration of the substitution $\mathtt{s}_v$.

$$\mathtt{Sub}_v = \{x \in \{\square, \blacksquare\}^{\mathbb{Z}^2} \mid \forall P \sqsubset x, \exists n \in \mathbb{N} : P \sqsubset \mathtt{s}_v^n(\blacksquare)\}$$

Proposition 3.2. *The following statements hold:*

(1) $\texttt{Sub}_v$ *is a minimal subshift.*

(2) $\texttt{s}_v$ *has unique derivation.*

(3) $\texttt{Sub}_v$ *is a* $\mathbb{Z}^2$*-sofic subshift which admits an almost 1-1 SFT extension*

(4) The Toeplitz configuration $\tau \in \{\square, \blacksquare\}^{\mathbb{Z}^2}$ *defined by:*

$$\tau_u = \begin{cases} \blacksquare \text{ if } \exists n \in \mathbb{N}, u \in p^n v + p^{n+1}\mathbb{Z}^2 \\ \square \text{ otherwise} \end{cases}$$

belongs to $\texttt{Sub}_v$*. In particular* $\overline{Orb(\tau)} = \texttt{Sub}_v$ *is a Toeplitz subshift.*

Proof. The substitution $\texttt{s}_v$ is primitive and any subshift generated by a primitive substitution is minimal, therefore (1) holds.

For the unique derivation, let $z, y \in \texttt{Sub}_v$ such that for any $u \in [0, p-1]^2 \cap \mathbb{Z}^2$ $z_{p(x,y)+u} = \texttt{s}_v(y_{(x,y)})_u$. As $(\texttt{s}_v(\square))_v = (\texttt{s}_v(\blacksquare))_v = \blacksquare$ we infer that $\forall w \in v + p\mathbb{Z}^2$ then $z_w = \blacksquare$. If there existed another way to subdivide z there would have to be a disjoint $p\mathbb{Z}^2$-lattice of $\blacksquare$. As each vector $u \in (\mathbb{Z}/p\mathbb{Z})^2 \setminus \{\vec{0}, v\}$ satisfies $(\texttt{s}_v(\square))_u = (\texttt{s}_v(\blacksquare))_u = \square$ the only possibility is that $\forall w \in \vec{0} + p\mathbb{Z}^2$ then $z_w = \blacksquare$, but as $(\texttt{s}_v(\square))_{\vec{0}} = \square$ this would imply that $y = \blacksquare^{\mathbb{Z}^2}$ which is clearly not an element of $\texttt{Sub}_v$. Hence (2) holds.

By Mozes Theorem [18] $\texttt{Sub}_v$ is a $\mathbb{Z}^2$-sofic subshift. Furthermore, as $\texttt{s}_v$ has unique derivation, the SFT extension can be chosen to be almost 1-1. This settles (3).

To show (4) it suffices to prove that all of the finite windows of the form $\tau|_{[0,p^n-1]^2}$ appear in some iteration of $\texttt{s}_v$. Indeed, as τ is Toeplitz, every pattern appearing in τ must also appear in the $\mathbb{N}^2$ portion of the plane. The claim follows directly as it can be verified inductively that $\tau|_{[0,p^n-1]^2} = \texttt{s}_v^n(\square)$ and $\texttt{s}_v^n(\square) \sqsubset \texttt{s}_v^{n+1}(\blacksquare)$. Also, as $\texttt{Sub}_v$ is minimal it follows that $\overline{\text{Orb}(\tau)} = \texttt{Sub}_v$ proving (4). □

Proposition 3.2 gives an useful way to describe elements from $\texttt{Sub}_v$. Namely, for each $z \in \texttt{Sub}_v$ there is a sequence of $\mathbb{Z}^2$ vectors $\{u_k\}_{k\in\mathbb{N}}$ such that $\sigma_{u_k}(\tau) \to z$. Let $K_n \in \mathbb{N}$ be such that $\forall k \geq K_n$ then $\sigma_{u_k}(\tau)|_{[0,p^n-1]^2} = z|_{[0,p^n-1]^2}$ and define $\bar{u}_n := u_{K_n} \mod (p^{n+1}, p^{n+1})$. We have that for $m \in \mathbb{N}$ $\sigma_{u_k}(\tau)|_{p^m v + u_k + p^{m+1}\mathbb{Z}^2}$ is composed uniquely of black squares. In the case where $m \leq n$ and $k \geq K_n$ we have

$$p^m v + u_k + p^{m+1}\mathbb{Z}^2 = \bar{u}_n + p^m v + p^{m+1}\mathbb{Z}^2.$$

Therefore, we can conclude that for every $m \leq n$ if we define $B_m(z) := \bar{u}_n + p^m v + p^{m+1}\mathbb{Z}^2$ then $z|_{B_m(z)}$ is composed uniquely of black squares. Moreover, each $\bar{u}_n$ is unique as any other possibility would shift the position of the $p^{m+1}\mathbb{Z}^2$-lattice of black squares which is already fixed.

Let $\{\bar{u}_n\}_{n\in\mathbb{N}}$ be the sequence of vectors associated to $z \in \texttt{Sub}_v$. Then for every $m \leq n$ we have $\bar{u}_m = \bar{u}_n \mod (p^{m+1}, p^{m+1})$. Conversely, for each sequence $\{\bar{u}_n\}_{n\in\mathbb{N}}$ which satisfies this restriction we can construct a configuration

$\bar{z} \in \texttt{Sub}_v$ as an accumulation point of $\sigma_{\bar{u}_n}(\tau)$ which therefore has the property that $B_n(\bar{z}) = \bar{u}_n + p^n v + p^{n+1}\mathbb{Z}^2$ for all $n \in \mathbb{N}$.

Proposition 3.3. *Let $\varphi \in Aut(\mathbb{Z}^2)$ be represented as $A_\varphi \in GL(2,\mathbb{Z})$ and let $A_{\widetilde{\varphi}} \in \mathcal{M}(\mathbb{Z}/p\mathbb{Z}, 2)$ be the matrix obtained by reducing the entries of A_φ modulo p. $A_{\widetilde{\varphi}}$ defines an automorphism $\widetilde{\varphi} \in Aut((\mathbb{Z}/p\mathbb{Z})^2)$ by left multiplication. We have that:*

$$\varphi(B_m(z)) = \varphi(\bar{u}_n) + p^m \widetilde{\varphi}(v) + p^{m+1}\mathbb{Z}^2.$$

In particular $\forall z \in \texttt{Sub}_v$ then $z \circ \varphi \in \texttt{Sub}_{\widetilde{\varphi}(v)}$ and $B_m(z \circ \varphi) = \varphi(B_m(z))$.

Proof. Let $z \in \texttt{Sub}_v$ and $B_m(z)$ as defined above, then, given any $n \geq m$:

$$\begin{aligned}
\varphi(B_m(z)) &= \varphi(\bar{u}_n + p^m v + p^{m+1}\mathbb{Z}^2) \\
&= \varphi(\bar{u}_n) + p^m A_\varphi v + p^{m+1}\mathbb{Z}^2 \\
&= \varphi(\bar{u}_n) + p^m (A_{\widetilde{\varphi}} + p(\frac{A_\varphi - A_{\widetilde{\varphi}}}{p}))v + p^{m+1}\mathbb{Z}^2 \\
&= \varphi(\bar{u}_n) + p^m A_{\widetilde{\varphi}} v + p^{m+1}((\frac{A_\varphi - A_{\widetilde{\varphi}}}{p})v + \mathbb{Z}^2) \\
&= \varphi(\bar{u}_n) + p^m \widetilde{\varphi}(v) + p^{m+1}\mathbb{Z}^2
\end{aligned}$$

This means that for fixed $n \in \mathbb{N}$ all lattices of size $m \leq n$ are sent to lattices appearing in configurations of $\texttt{Sub}_{\widetilde{\varphi}(v)}$. Indeed, as $\bar{u}_m = \bar{u}_n \mod (p^{m+1}, p^{m+1})$ we have $\varphi(\bar{u}_m) = \varphi(\bar{u}_n) \mod (p^{m+1}, p^{m+1})$ and therefore the sequence $\{\varphi(\bar{u}_n)\}_{n \in \mathbb{N}}$ defines a configuration in $\texttt{Sub}_{\widetilde{\varphi}(v)}$. Following a compactness argument one concludes that $z \circ \varphi \in \texttt{Sub}_{\widetilde{\varphi}(v)}$ and $B_m(z \circ \varphi) = \varphi(B_m(z))$. □

The importance of Proposition 3.3 is that it states that any automorphism of $\mathbb{Z}^2$ correctly maps the lattices $B_m(z)$ to those of another substitution. We shall use these lattices to encode elements of $\{0,1\}^{\mathbb{N}}$ belonging to our H-dynamical system (X, f). In order to do this, we need to define a subshift which matches these lattices to actual values from X and that also codes the action of f.

3.2 Encoding configurations in Toeplitz sequences.

Consider $p \geq 3, q \in \{1, \dots, p-1\}$ and the encoding $\Psi_q : \{0,1\}^{\mathbb{N}} \to \{0,1,\$\}^{\mathbb{Z}}$ given by:

$$\Psi_q(x)_j = \begin{cases} x_n & \text{if } j = qp^n \mod p^{n+1} \\ \$ & \text{in the contrary case.} \end{cases}$$

The idea behind this encoding is to match for each $m \in \mathbb{N}$ the horizontal and vertical projections of the lattice $B_m(z)$ for some $z \in \texttt{Sub}_v$ to the symbol x_m. We need to do this for every possible choice of q as the projections of the lattices might differ depending on the substitution. For instance, the horizontal projection associated to $v = (1,1)$ is different from the one for $v = (2,2)$. We begin this section by studying the structure of the encoding Ψ_q.

First notice that $\Psi_q(x)|_{q+p\mathbb{Z}} \equiv x_0$ and $\forall q' \in \{1, \dots p-1\} \setminus \{q\}$ we have that $\Psi_q(x)_{q'+p\mathbb{Z}} \equiv \$$. Indeed, as $q' + pk \neq 0 \mod p$ thus $q' + pk \neq p^i \mod p^{i+1}$. Also, if $i \geq 1$ and $\Psi_q(x)_j = x_i$ then $\Psi_q(x)_{j+q} = x_0$ as $j = p^i \mod p^{i+1}$ forces that $j = 0 \mod p$. This means that every x_0 is a special coordinate in a string of $p-1$ symbols where every other symbol is \$ and every x_i with $i \geq 1$ is necessarily followed by such string. The important property we derive from these computations is that the lattice of x_0 can be recognized locally. Indeed, each x_0 is preceded by $q-1$ symbols \$ and followed by $p-q-1$ symbols \$. If $p \geq 3$ and $q-1 \neq p-q-1$ this is enough to locally recognize the position of the lattice in a string of p symbols as x_0 is the only lattice satisfying that property. If $q-1 = p-q-1$ the previous property is now true for any symbol but the decoding can be done in any string of $2p$ symbols because if $\Psi_q(x)_j = x_m$ for some $m > 0$ then $\Psi_q(x)_{j+p} = \Psi_q(x)_{j-p} = \$$ and any false positive can be detected in a finite window.

For $x = (x_i)_{i\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ let $\sigma(x) \in \{0,1\}^{\mathbb{N}}$ be defined by $\sigma(x)_i = x_{i+1}$ (we shall use the same notation as in the case of the group shift action, though in this case it's a one-sided $\mathbb{N}$-action). We define also for $k \in \mathbb{Z}/p\mathbb{Z}$ the transformation $\Omega_k : \{0,1,\$\}^{\mathbb{Z}} \to \{0,1,\$\}^{\mathbb{Z}}$ by $(\Omega_k(y))_j = y_{jp+k}$. It is straightforward from the definition of Ψ_q that Ω_0 transforms the coding of $x \in \{0,1\}^{\mathbb{N}}$ into that of its shifted version, that is $\Omega_0(\Psi_q(x)) = \Psi_q(\sigma(x))$. Also, one can directly check that $\Omega_k = \Omega_0 \circ \sigma_{-k}$ and $\Omega_0 \circ \sigma_{p\ell} = \sigma_\ell \circ \Omega_0$.

Proposition 3.4. *Let $x \in \{0,1\}^{\mathbb{N}}$ and $y \in \overline{Orb_\sigma(\Psi_q(x))}$. There exists an unique $k_0 \in \mathbb{Z}/p\mathbb{Z}$ such that:*

$$\Omega_{k_0}(y) \in \overline{Orb_\sigma(\Psi_q(\sigma(x)))}.$$

Proof. The application Ω_k is clearly continuous in the product topology as fixing y in the interval $\mathbb{Z} \cap [-lp, lp-1]$ for $l \geq 1$ necessarily fixes $\Omega_k(y)$ in the interval $\mathbb{Z} \cap [-l, l-1]$.

Let $y \in \overline{\text{Orb}_\sigma(\Psi_q(x))}$. As $\Psi_q(x)|_{q+p\mathbb{Z}} \equiv x_0$ we can deduce by compactness that there exists $k \in \mathbb{Z}/p\mathbb{Z}$ such that $y|_{k+p\mathbb{Z}} \equiv x_0$. Define $k_0 := k - q \mod p$. Then for each $n \in \mathbb{Z}$ we have $y_{pn+k_0+1}, \dots, y_{k_0+p-1} = \$^{q-1}x_0\$^{p-q-1}$ as words. This necessarily implies that any other choice of k_0 would make $\Omega_{k_0}(y)$ be a constant configuration which clearly does not belong to $\overline{\text{Orb}_\sigma(\Psi_q(\sigma(x)))}$ therefore making the previous choice the only possible one. Consider a sequence $(\sigma_{n_i}(\Psi_q(x)))_{i\in\mathbb{N}} \to y$. Without loss of generality we can ask that $n_i \in p\mathbb{Z} + k_0$, if not it suffices to eliminate a finite number of terms. For any n_i of the form $p\ell + k_0$ we get that

$$\begin{aligned}
\Omega_{k_0}(\sigma_{p\ell+k_0}(\Psi_q(x))) &= \Omega_0 \circ \sigma_{-k_0} \circ \sigma_{k_0} \circ \sigma_{p\ell}(\Psi_q(x)) \\
&= \Omega_0 \circ \sigma_{p\ell}(\Psi_q(x)) \\
&= \sigma_\ell \circ \Omega_0(\Psi_q(x)) \\
&= \sigma_\ell(\Psi_q(\sigma(x))) \in \text{Orb}(\Psi_q(\sigma(x)))
\end{aligned}$$

As Ω_k is continuous, we obtain that $\Omega_{k_0}(y) \in \overline{\text{Orb}_\sigma(\Psi_q(\sigma(x)))}$. □

Example. *For $p = 3$, $q = 1$ and $x = x_0x_1x_2\dots$ we obtain that:*

$$\Psi_q(x)|_{\{0,\dots,30\}} = \$x_0\$x_1x_0\$\$x_0\$x_2x_0\$x_1x_0\$\$x_0\$\$x_0\$x_1x_0\$\$x_0\$x_3x_0\$x_1$$
$$\Omega_0(\Psi_q(x))|_{\{0,\dots,10\}} = \$x_1\$x_2x_1\$\$x_1\$x_3x_1 = \Psi_q(\sigma(x))|_{\{0,\dots,10\}}$$
$$\Omega_0^2(\Psi_q(x))|_{\{0,\dots,3\}} = \$x_2\$x_3 = \Psi_q(\sigma^2(x))|_{\{0,\dots,3\}}$$

Proposition 3.4 shows explicitly that $x \in \{0,1\}^{\mathbb{N}}$ can be decoded not only from $\Psi_q(x)$ but from any element of the closure of the orbit of $\Psi_q(x)$ under the shift action. Indeed, given $y^0 \in \overline{\mathrm{Orb}_\sigma(\Psi_q(x))}$ we find the unique value $k_0 \in \mathbb{Z}/p\mathbb{Z}$ as above and deduce that $x_0 = (y^0)_{k_0+q}$. Next one takes $y^1 := \Omega_{k_0}(y^0)$ and finds a new value k_1 as before and so $x_1 = (y^1)_{k_1+q}$. Iterating this procedure one gets a sequence y^i such that $y^i := \Omega_{k_i}(y^{i-1}) \in \overline{\mathrm{Orb}_\sigma(\Psi_q(\sigma^i(x)))}$ and $x_i = (y^i)_{k_i+q}$.

Proposition 3.5. *Let $x, x' \in \{0,1\}^{\mathbb{N}}$. $\overline{Orb_\sigma(\Psi_q(x))} \cap \overline{Orb_\sigma(\Psi_q(x'))} \neq \emptyset$ if and only if $x = x'$.*

Proof. Let $y \in \overline{\mathrm{Orb}_\sigma(\Psi_q(x))} \cap \overline{\mathrm{Orb}_\sigma(\Psi_q(x'))}$. Using Proposition 3.4 we can find an unique $k_0 \in \mathbb{Z}/p\mathbb{Z}$ such that $\Omega_{k_0}(y) \in \overline{\mathrm{Orb}_\sigma(\Psi_q(\sigma(x)))}$ and thus $x_0 = y_{k_0+q}$. The same proposition gives $k_0' \in \mathbb{Z}/p\mathbb{Z}$ such that $\Omega_{k_0'}(y) \in \overline{\mathrm{Orb}_\sigma(\Psi_q(\sigma(x')))}$ and so $x_0' = y_{k_0'+q}$. Nevertheless in the proof of Proposition 3.4 we see that any other choice of k_0 would give a constant configuration and therefore $k_0 = k_0'$. This implies that $x_0 = x_0'$ and $\Omega_{k_0}(y) \in \overline{\mathrm{Orb}_\sigma(\Psi_q(\sigma(x)))} \cap \overline{\mathrm{Orb}_\sigma(\Psi_q(\sigma(x')))}$. Iterating this argument we obtain that for every $n \in \mathbb{N}$ then $x_n = x_n'$ holds and thus $x = x'$. The other direction is trivial as $\Psi_q(x) \in \overline{\mathrm{Orb}_\sigma(\Psi_q(\sigma(x)))} \neq \emptyset$. □

Before continuing, let's draw the attention to the structure of the subshift $\overline{\mathrm{Orb}_\sigma(\Psi_q(x))}$. Every element here encodes the structure of x by repeating its n-th coordinate in gaps of size p^{n+1}. Therefore, every non \$ element appears periodically with at most one exception – a position obtained by compactness – which we denote by x_∞. This point may take any value if both 0 and 1 appear infinitely often in x but is restricted if x is eventually constant. This point has its analogue in the configurations $z \in \mathtt{Sub}_v$. All of the ■ symbols appear in square lattices with the exception of at most one. We call this degenerated lattice $B_\infty(z)$ and make the remark that $B_\infty(z)$ might be empty.

Let (X, f) be an H-dynamical system and $p \geq 3$. We use the encoding Ψ_q defined above to construct a $\mathbb{Z}$-subshift $\mathtt{Top}(X, f)$ which encodes the configurations of X and the action of f around a unit ball in H. Formally, let $S \subset H$ be a finite set such that $1_H \in S$ and $\langle S \rangle = H$. We define by $\Psi(x)$ as the configuration in $(\{0,1,\$\}^{\{1,\dots,p-1\}\times S})^{\mathbb{Z}}$ such that $(\Psi(x)_n)_{(q,s)} = \Psi_q(f_s(x))_n$.

$\mathtt{Top}(X, f) \subset (\{0,1,\$\}^{\{1,\dots,p-1\}\times S})^{\mathbb{Z}}$ is the $\mathbb{Z}$-subshift given by:

$$\mathtt{Top}(X, f) := \bigcup_{x \in X} \left(\overline{\mathrm{Orb}_\sigma\left(\Psi(x)\right)} \right)$$

Elements of $\mathtt{Top}(X, f)$ can be thought of as $(p-1)|S|$-tuples of configurations in $\{0,1,\$\}^{\mathbb{Z}}$ where the tuple associated to the pair (q, s) belongs to

$\overline{\mathrm{Orb}_\sigma(\Psi_q(f_s(x)))}$. Here the shift action is taken diagonally, that is, each configuration is shifted simultaneously. The idea behind this construction is to let each q-row code an element $x \in X$ and its image $f_s(x)$ for each $s \in S$. Given $y \in \texttt{Top}(X, f)$ we denote the projection to the (q, s)-th layer by $\texttt{Layer}_{q,s}(y) \in \{0, 1, \$\}^{\mathbb{Z}}$. We need to do this for every possible q just for technical reasons, as we'll need to match every possible lattice in the substitution defined above.

Proposition 3.6. *If (X, f) is an effectively closed H-dynamical system then $\texttt{Top}(X, f)$ is an effectively closed $\mathbb{Z}$-subshift.*

Proof. $\texttt{Top}(X, f)$ is clearly shift invariant. To see that it is closed consider a sequence of configurations $\{y_n\}_{n\in\mathbb{N}} \subset \texttt{Top}(X, f)$ converging to y. By Proposition 3.5 each y_n belongs to a unique orbit $\overline{\mathrm{Orb}_\sigma(\Psi(x^n))}$ for $x^n \in X$ as they are pairwise disjoint. It is also straightforward to see that $y \in \overline{\mathrm{Orb}_\sigma(\Psi(x))}$ for some $x \in \{0, 1\}^{\mathbb{N}}$. It suffices to show that $x \in X$. As Ω_k is continuous we get that $\Omega_k(\texttt{Layer}_{q,s}(y_n)) \to \Omega_k(\texttt{Layer}_{q,s}(y))$. Clearly the sequence of k_0 given by Proposition 3.4 associated to y_n must stabilize and hence there is $N \in \mathbb{N}$ such that for every $n \geq N$ then y_n belongs to an orbit $\overline{\mathrm{Orb}_\sigma(\Psi(x^n))}$ where $(x^n)_0 = x_0 = (\texttt{Layer}_{q,1_H}(y))_{k_0+q}$. Iterating this argument we get that for each $m \in \mathbb{N}$ there exists N_m such that for every $n \geq N_m$ then $(x^n)_i = x_i$ for each $i \leq m$. We conclude that x^n converges to x. As X is closed we obtain that $y \in \texttt{Top}(X, f)$.

A set of forbidden patterns defining $\texttt{Top}(X, f)$ can be given explicitly. We consider for $n \in \mathbb{N}$ all words of length p^{n+1} over the alphabet $\{\$, 0, 1\}^{|S|(p-1)}$ which do not appear in any configuration of $\texttt{Top}(X, f)$. As this is an increasing sequence of forbidden patterns it is enough to define $\texttt{Top}(X, f)$.

This set of forbidden words is recursively enumerable. The following algorithm accepts a set of forbidden patterns defining $\texttt{Top}(X, f)$. Let the input be a word of length p^n for $n \in \mathbb{N}$. The structure of $\texttt{Top}(X, f)$ makes it possible to recognize algorithmically all gaps in every layer (formally the algorithm checks that each substring of p contiguous symbols is a cyclic permutation of $a\$^{q-1}b\$^{p-q-1}$ for some $a \in \{0, 1, \$\}$ and $b \in \{0, 1\}$). Then if this stage is passed, it computes k_0 from Proposition 3.4 for each layer, checks that b is the same symbol throughout the word. Finally it checks that k_0 is the same in every layer (thus the layers are aligned). Then it applies Ω_{k_0} to this string obtaining a word of length p^{n-1}. The algorithm is repeated until reaching a word of length 0. If at any stage a check fails, the word is accepted as forbidden.

The previous stage recognizes all words that haven't got the correct structure. After that stage ends, we can use the same algorithm and the function Ω_k to decode n coordinates $x_0x_1 \dots x_{n-1}$ for each pair (q, s) and check for every s that the word is the same independently of q. If this stage is passed we end up with $|S|$ words which depend only on s and we denote them by $(w_s)_{s\in S}$. Here we run two recognition algorithms in parallel. One searches for a cylinder $[w_s] \not\subset X$ and the other searches if $[w_{1_H}] \not\subset f_s^{-1}([w_s])$. If any of these two searches succeed at a certain step then the algorithm returns that the pattern is forbidden. These two last algorithms do exist as (X, f) is an effectively closed H-dynamical system. □

The subshift $\texttt{Top}(X, f)$ is the ingredient of the proof which allows us to simulate points $x \in X$ and their images under the generators of H in a sofic $\mathbb{Z}^2$-subshift which contains this information. The next step is to put one of these configurations in each $\mathbb{Z}^2$-coset of $\mathbb{Z}^2 \rtimes_\varphi H$ and force by local rules that the shift action by $(0, h)$ yields the $\mathbb{Z}^2$-coset where the point $f_h(x)$ is codified. The obvious obstruction to this idea is the fact that the action under $(0, h)$ in a semidirect product disturbs the adjacency relations in a coset if the automorphism φ_h isn't trivial. The way to go around this obstruction is to use the lattices given by the layer $\texttt{Sub}_v$ which are invariant under automorphisms. We specify how these two elements go together in the next subsection.

3.3 Proof of Theorem 3.1

Denote $\varphi : H \to \mathrm{Aut}(\mathbb{Z}^2)$ a group homomorphism such that $G = \mathbb{Z}^2 \rtimes_\varphi H$ is given by:

$$(n_1, h_1) \cdot (n_2, h_2) = (n_1 + \varphi_{h_1}(n_2), h_1 h_2)$$

Let S be a finite set of generators of H where $1_H \in S$, $|S| = d$ and let's fix the parameter $p = 3$ which is used to construct $\texttt{Top}(X, f)$ (which contains thus $2d$ layers) and the substitutions $\texttt{Sub}_v$ for $v \in (\mathbb{Z}/3\mathbb{Z})^2 \setminus \{\vec{0}\}$. Consider the following two $\mathbb{Z}^2$-subshifts.

$$\texttt{Top}(X, f)^H \subseteq (\{0, 1, \$\}^{2d})^{\mathbb{Z}^2}$$
$$\texttt{Top}(X, f)^V \subseteq (\{0, 1, \$\}^{2d})^{\mathbb{Z}^2}$$

Where $x \in \texttt{Top}(X, f)^H$ is the subshift whose projection to $(\mathbb{Z}, 0)$ belongs to $\texttt{Top}(X, f)$ and any vertical strip is constant. Analogously $x \in \texttt{Top}(X, f)^V$ is the subshift whose projection to $(0, \mathbb{Z})$ belongs to $\texttt{Top}(X, f)$ and any horizontal strip is constant. Formally: $x \in \texttt{Top}(X, f)^H$ if $\forall i, j \in \mathbb{Z}$ then $x_{i,j} = x_{i,j+1}$ and $(x_{(i,0)})_{i \in \mathbb{Z}} \in \texttt{Top}(X, f)$. An analogous definition can be given for $\texttt{Top}(X, f)^V$. Proposition 3.6 says that $\texttt{Top}(X, f)$ is an effective $\mathbb{Z}$-subshift and therefore $\texttt{Top}(X, f)^H$ and $\texttt{Top}(X, f)^V$ are sofic $\mathbb{Z}^2$-subshifts by the simulation theorem proven in [3, 9]. Next we are going to put these subshifts together with the substitution layers to create a rich structure in each $\mathbb{Z}^2$-coset.

Consider the product subshift $\texttt{Top}(X, f)^H \times \texttt{Top}(X, f)^V \times \bigotimes_{v \in (\mathbb{Z}/3\mathbb{Z})^2 \setminus \{\vec{0}\}} \texttt{Sub}_v$. Given a configuration y in that product we denote by $\texttt{Layer}^H(y)$ and $\texttt{Layer}^V(y)$ the projections to the horizontal and vertical layer respectively. If we want to precise the tuple we will use the notation $\texttt{Layer}^H_{q,s}(y)$ and $\texttt{Layer}^V_{q,s}(y)$ respectively. Also, for $v \in (\mathbb{Z}/3\mathbb{Z})^2 \setminus \{\vec{0}\}$, we denote by $\texttt{Sub}_v(y)$ the projection to the corresponding substitutive layer. We define $\Pi(X, f) \subset \texttt{Top}(X, f)^H \times \texttt{Top}(X, f)^V \times \bigotimes_{v \in (\mathbb{Z}/3\mathbb{Z})^2 \setminus \{\vec{0}\}} \texttt{Sub}_v$ as the set of configurations y which satisfy the following rules:

1. $\forall u \in \mathbb{Z}^2$ and $(a, b) \in (\mathbb{Z}/3\mathbb{Z})^2 \setminus \{\vec{0}\}$ the following is satisfied. If $a \neq 0$ then $(\texttt{Sub}_{(a,b)}(y))_u = \blacksquare$ if and only if $(\texttt{Layer}^H_{a,1_H}(y))_u \in \{0, 1\}$. Analogously, if $b \neq 0$ then then $(\texttt{Sub}_{(a,b)}(y))_u = \blacksquare$ if and only if $(\texttt{Layer}^V_{b,1_H}(y))_u \in \{0, 1\}$.

2. If $(\texttt{Sub}_{(1,1)}(y))_u = \blacksquare$ then $\forall s \in S$ $(\texttt{Layer}^H_{1,s}(y))_u = (\texttt{Layer}^V_{1,s}(y))_u$.

The $\mathbb{Z}^2$-subshift $\Pi(X,f)$ is sofic. Indeed, all the component are sofic subshifts and the added rules are local (we just forbid symbols in the product alphabet). Recall that we denote by $B_m(z)$ the m-th lattice of black squares in a configuration z in a substitutive layer.

Claim 3.1. *Let $y \in \Pi(X,f)$, $(a,b) \in (\mathbb{Z}/3\mathbb{Z})^2 \setminus \{\vec{0}\}$ and $z = \texttt{Sub}_{(a,b)}(y)$. Suppose that $\texttt{Layer}^H(y)$ is given by $x \in X$. Then:*

- *If $a \neq 0$ then $\forall m \in \mathbb{N}, \forall s \in S$: $\texttt{Layer}^H_{a,s}(y)|_{B_m(z)} \equiv f_s(x)_m$*
- *If $b \neq 0$ then $\forall m \in \mathbb{N}, \forall s \in S$: $\texttt{Layer}^V_{b,s}(y)|_{B_m(z)} \equiv f_s(x)_m$*
- *The configurations in the layers $\texttt{Top}(X,f)^H$ and $\texttt{Top}(X,f)^V$ are defined by the same $x \in X$.*

Proof. Let $a \neq 0$. It suffices to show this property for $s = 1_H$ as the definition of $\texttt{Top}(X,f)$ forces the configurations to be aligned. The lattice $B_0(z)$ has the form $\bar{u}_0 + (a,b) + 3\mathbb{Z}^2$, therefore its projection in the horizontal coordinate is of the form $k_0 + 3\mathbb{Z}$. Using the structure of $\Psi_a(x)$ there are three possibilities for 3-lattices: One contains uniformly the symbol x_0, another contains only the symbol \$ and the third one contains $\Psi_a(\sigma(x))$ by proposition 3.4. The first rule of $\Pi(X,f)$ rules out the second and third possibility because there would be \$'s matched with $\blacksquare$. Therefore $\texttt{Layer}^H_{a,1_H}|_{B_0(z)} \equiv x_0$. Inductively, let $B_m(z) = \bar{u}_m + (a,b)3^m + 3^{m+1}\mathbb{Z}^2$ and suppose $\forall m' < m$ $\texttt{Layer}^H_{a,1_H}|_{B_{m'}(z)} \equiv x_{m'}$. Note that for m' the projection to the horizontal layer is of the form $k_{m'} + 3^{m'+1}\mathbb{Z}$. Using iteratively the previous argument and applying the function $\Omega_{k_{m'}}$ defined in 3.4 we end up with three possibilities for 3^m-lattices (that is, the value of $k_{m'}$), and again the first rule of $\Pi(X,f)$ rules out two of them, yielding $\texttt{Layer}^H_{a,1_H}|_{B_m(z)} \equiv x_m$.

Suppose the configuration in $\texttt{Top}(X,f)^V$ is given by $x' \in X$. For b the proof is analogous and we get that $b \neq 0$ implies that $\forall m \in \mathbb{N}, \forall s \in S$: $\texttt{Layer}^V_{b,s}|_{B_m(z)} \equiv f_s(x')_m$.

Now set $(a,b) = (1,1)$. The second rule of $\Pi(X,f)$ implies that $\forall s \in S, m \in \mathbb{N}$ then $(\texttt{Layer}^H_{1,s}(y))|_{B_m(z)} = (\texttt{Layer}^V_{1,s}(y))|_{B_m(z)}$. Using the previous two properties we conclude that $\forall s \in S, m \in \mathbb{N}$ we have $f_s(x)_m = f_s(x')_m$. Using $s = 1_H$ yields $x = x'$ hence proving the second and third statement. □

From Claim 3.1 we obtain that each configuration $y \in \Pi(X,f)$ contains the information of a single $x \in X$. We can thus define properly the decoding function $\Upsilon : \Pi(X,f) \to X$ such that $\Upsilon(y) = x$ if and only if $\forall m \in \mathbb{N}$: $\texttt{Layer}^H_{1,1_H}(y)|_{B_m(\texttt{Sub}_{(1,1)}(y))} \equiv x_m$.

Consider the set of forbidden patterns $\mathcal{F}$ defining $\Pi(X,f)$. Each of these patterns has a finite support $F \subset \mathbb{Z}^2$. We extend those patterns to patterns in $G = \mathbb{Z}^2 \rtimes_\varphi H$ by associating $d \in F \to (d, 1_H) \in G$. Therefore every pattern $P \in \mathcal{F}$ with support $F \subset \mathbb{Z}^2$ is embedded into a pattern $\widetilde{p}$ with support $F \times \{1_H\} \subset G$. We consider the set $\widetilde{\mathcal{F}} = \{\widetilde{p} \mid p \in \mathcal{F}\}$ and we define $\texttt{Final}(X,f)$ as the subshift over the same alphabet as $\Pi(X, f)$ defined by the set of forbidden patterns $\widetilde{\mathcal{F}} \cup \mathcal{G}$ where $\mathcal{G}$ is defined as follows:

For each $s \in S$ consider $\varphi_{s^{-1}}$ the automorphism associated to s^{-1} and $(a, b) = \widetilde{\varphi}_{s^{-1}}(1, 1)$. We put in $\mathcal{G}$ all the patterns P with support $\{(\vec{0}, 1_H), (\vec{0}, s^{-1})\}$ which satisfy that $\texttt{Sub}_{(a,b)}(P_{(\vec{0},1_H)}) = \blacksquare$ but either:

- $\texttt{Sub}_{(1,1)}(P_{(\vec{0},s^{-1})}) \neq \blacksquare$ or
- $\texttt{Sub}_{(1,1)}(P_{(\vec{0},s^{-1})}) = \blacksquare$ and
  - If $a \neq 0$ then $\texttt{Layer}^H_{a,s}(P_{(\vec{0},1_H)}) \neq \texttt{Layer}^H_{1,1_H}(P_{(\vec{0},s^{-1})})$ or
  - If $b \neq 0$ then $\texttt{Layer}^V_{b,s}(P_{(\vec{0},1_H)}) \neq \texttt{Layer}^V_{1,1_H}(P_{(\vec{0},s^{-1})})$.

In simpler words: we force that every $\blacksquare$ in layer $\texttt{Sub}_{(a,b)}$ of the $(\mathbb{Z}^2, 1_H)$-coset must be matched with a $\blacksquare$ in $\texttt{Sub}_{(1,1)}$ in the $(\mathbb{Z}^2, s^{-1})$-coset and that if $a \neq 0$ then the symbol in $(\vec{0}, 1_H)$ in $\texttt{Layer}^H_{a,s}$ is the same as the symbol in $(\vec{0}, s^{-1})$ in $\texttt{Layer}^H_{1,1_H}$. If $b \neq 0$ we impose that the symbol in $(\vec{0}, 1_H)$ in $\texttt{Layer}^V_{b,s}$ is the same as the symbol in $(\vec{0}, s^{-1})$ in $\texttt{Layer}^V_{1,1_H}$.

Before continuing let's translate $\widetilde{\mathcal{F}} \cup \mathcal{G}$ into properties of $\texttt{Final}(X, f)$. In order to do that properly, for $y \in \texttt{Final}(X, f)$ we denote by $\pi(y)$ the $\mathbb{Z}^2$-configuration such that $\forall u \in \mathbb{Z}^2$ $\pi(y)_u = y_{(u,1_H)}$.

Claim 3.2. *$\texttt{Final}(X, f)$ satisfies the following properties:*

- *$\texttt{Final}(X, f)$ is a sofic G-subshift.*
- *Let $y \in \texttt{Final}(X, f)$. Then $\pi(y) \in \Pi(X, f)$.*
- *If $\Upsilon(\pi(y)) = x$ then $\forall h \in H$, $\Upsilon(\pi(\sigma_{(\vec{0},h)}(y))) = f_h(x)$.*

Proof. As $\Pi(X, f)$ is sofic, it admits an SFT extension $\phi : \widehat{\Pi}(X, f) \twoheadrightarrow \Pi(X, f)$. By embedding as above a finite list of forbidden patterns defining $\widehat{\Pi}(X, f)$ into G we obtain a G-SFT extension of $X_{\widetilde{\mathcal{F}}}$. Adding to this list of forbidden patterns the pullback of the finite list of forbidden patterns $\mathcal{G}$ under the local code Φ defining ϕ we obtain an SFT extension $\widehat{\texttt{Final}}(X, f)$ of $\texttt{Final}(X, f)$.

The second property comes directly from the definition of $\texttt{Final}(X, f)$ as it contains an embedding of every forbidden pattern defining $\Pi(X, f)$. Note that it may happen that $y|_{(\mathbb{Z}^2,h)}$ seen as a $\mathbb{Z}^2$-configuration does not belong to $\Pi(X, f)$ for some $h \in H$, but $\pi(\sigma_{(\vec{0},h)}(y))$ always does.

Let's prove the third property: We claim that it suffices to prove the property for $s \in S$. Indeed, given $h \in H$, as $H = \langle S \rangle$ there exists a minimal length word representing h. If $h = 1_H$ the result is immediate. If not, then $h = sh'$ for some $h' \in H$ having a shorter word representation. Suppose this third property holds for all words of strictly smaller length and define $y' = \sigma_{(\vec{0},h')}(y)$. We have that $\Upsilon(\pi(y')) = f_{h'}(x) = x'$, so:

$$\Upsilon(\pi(\sigma_{(\vec{0},h)}(y))) = \Upsilon(\pi(\sigma_{(\vec{0},s)}(y'))) = f_s(x') = f_s(f_{h'}(x)) = f_h(x).$$

It suffices therefore to prove the property for $s \in S$. Let's denote $y' = \sigma_{(\vec{0},s)}(y)$ and let $\Upsilon(\pi(y)) = x$ and $\Upsilon(\pi(y')) = x'$. We want to prove that $x' = f_s(x)$. Let $\widetilde{\varphi}_{s^{-1}}(1,1) = (a,b)$ and suppose that $a \neq 0$ (if $a = 0$ then $b \neq 0$ and the argument is analogous). Let $m \in \mathbb{N}$, using Claim 3.1 we obtain

$$\texttt{Layer}^H_{1,1_H}(y')|_{(B_m(\texttt{Sub}_{(1,1)}(y')),1_H)} \equiv x'_m$$

$$\texttt{Layer}^H_{a,s}(y)|_{(B_m(\texttt{Sub}_{(a,b)}(y)),1_H)} \equiv f_s(x)_m.$$

Using the forbidden patterns $\mathcal{G}$ results in

$$\texttt{Sub}_{(1,1)}(y)|_{(B_m(\texttt{Sub}_{(a,b)}(y)),s^{-1})} \equiv \blacksquare$$

$$\texttt{Layer}^H_{1,1_H}(y)|_{(B_m(\texttt{Sub}_{(a,b)}(y)),s^{-1})} \equiv f_s(x)_m.$$

Finally, developing the action on y' yields

$$\begin{aligned} y'|_{(B_m(\texttt{Sub}_{(1,1)}(y')),1_H)} &= \sigma_{(\vec{0},s)}(y)|_{(B_m(\texttt{Sub}_{(1,1)}(y')),1_H)} \\ &= y|_{(\vec{0},s^{-1})(B_m(\texttt{Sub}_{(1,1)}(y')),1_H)} \\ &= y|_{(\varphi_{s^{-1}}(B_m(\texttt{Sub}_{(1,1)}(y'))),s^{-1})}. \end{aligned}$$

Using Proposition 3.3 we obtain that:

$$\varphi_{s^{-1}}(B_m(\texttt{Sub}_{(1,1)}(y'))) = B_m(\texttt{Sub}_{(1,1)}(y')\circ\varphi_{s^{-1}}) \text{ and } \texttt{Sub}_{(1,1)}(y')\circ\varphi_{s^{-1}} \in \texttt{Sub}_{(a,b)}.$$

As we also have $\forall m \in \mathbb{N}$ that:

$$\texttt{Sub}_{(1,1)}(y')|_{(B_m(\texttt{Sub}_{(1,1)}(y')),1_H)} \equiv \blacksquare \text{ and } \texttt{Sub}_{(1,1)}(y)|_{(B_m(\texttt{Sub}_{(a,b)}(y)),s^{-1})} \equiv \blacksquare$$

we conclude that $\varphi_{s^{-1}}(B_m(\texttt{Sub}_{(1,1)}(y'))) = B_m(\texttt{Sub}_{(a,b)}(y))$. Therefore,

$$\texttt{Layer}^H_{1,1_H}(y')|_{(B_m(\texttt{Sub}_{(1,1)}(y')),1_H)} = \texttt{Layer}^H_{1,1_H}(y)|_{(B_m(\texttt{Sub}_{(a,b)}(y)),s^{-1})}.$$

Which yields $x'_m = f_s(x)_m$. As $m \in \mathbb{N}$ is arbitrary $x' = f_s(x)$. □

Finally we are ready to finish the proof. Consider again the SFT extension $\widehat{\texttt{Final}}(X,f)$ of $\texttt{Final}(X,f)$, the factor map $\phi : \widehat{\texttt{Final}}(X,f) \twoheadrightarrow \texttt{Final}(X,f)$ and the subaction $(\widehat{\texttt{Final}}(X,f), \sigma^H)$.

Proposition 3.7. *$\Upsilon \circ \pi \circ \phi$ is a factor map from $(\widehat{\texttt{Final}}(X,f), \sigma^H)$ to (X,f).*

Proof. As $\phi : \widehat{\texttt{Final}}(X,f) \twoheadrightarrow \texttt{Final}(X,f)$ it suffices to show that $\Upsilon \circ \pi$ is a factor map from $(\texttt{Final}(X,f), \sigma^H)$ to (X,f). Let $y \in \texttt{Final}(X,f)$. Following Claim 3.2 we have $\pi(y) \in \Pi(X,f)$ and thus $\Upsilon(\pi(y)) \in X$. Moreover, setting $\Upsilon(\pi(y)) = x$ yields $\forall h \in H$ that $\Upsilon(\sigma_{(\vec{0},h)}(y)) = f_h(x)$. This implies

$$\forall h \in H : (\Upsilon \circ \pi) \circ \sigma_{(\vec{0},h)} = f_h \circ (\Upsilon \circ \pi).$$

Also, both Υ and π are clearly continuous, therefore, it only remains to show that $\Upsilon\circ\pi$ is surjective. Let $x \in X$, we construct a configuration $\hat{y} \in \texttt{Final}(X, f)$ such that $\Upsilon(\pi(\hat{y})) = x$.

In order to do this, we begin by constructing a sequence of configurations $(y^h)_{h\in H}$ which belong to $\Pi(X, f)$. For $(a, b) \in (\mathbb{Z}/3\mathbb{Z})^2 \setminus \{\vec{0}\}$ let $z_{(a,b)} \in \texttt{Sub}_{(a,b)}$ be the Toeplitz configuration defined in Proposition 3.2 part (4). The configuration $z_{(a,b)}$ satisfies $B_m(z_{(a,b)}) = (a, b)3^m+3^{m+1}\mathbb{Z}^2$ for $m \in \mathbb{N}$ and $B_\infty(z_{(a,b)}) = \emptyset$. We define $y^h \in \Pi(X, f)$ by specifying the configuration in each layer. For a substitutive layer we have $\texttt{Sub}_{(a,b)}(y^h) = z_{(a,b)}$ and for the Toeplitz layers we have that $\forall u = (u_1, u_2) \in \mathbb{Z}^2$, $s \in S$, $a, b \in \{1, 2\}$ then $\texttt{Layer}^H_{a,s}(y^h)_u = \Psi_a(f_s(f_h(x)))_{u_1}$ and $\texttt{Layer}^V_{b,s}(y^h)_u = \Psi_b(f_s(f_h(x)))_{u_2}$. It can easily be verified that for each $h \in H$ the configuration $y^h \in \Pi(X, f)$.

Finally, we define $\hat{y}$ as follows:

$$\hat{y}_{(u,h)} = (y^{h^{-1}})_{\varphi_{h^{-1}}(u)}.$$

As $\varphi_{1_H} u = u$ then $\pi(\hat{y}) = y^{1_H}$ and thus $\Upsilon(\pi(\hat{y})) = f_{1_H}(x) = x$. It suffices to show that $\hat{y} \in \texttt{Final}(X, f)$. This comes down to showing that no patterns in $\mathcal{F}$ or $\mathcal{G}$ appear in $\hat{y}$. Suppose a pattern $P \in \mathcal{F}$ appears at position $g = (u, h)$, that is $\hat{y} \in [P]_g \iff \sigma_{g^{-1}}(\hat{y}) \in [P]_{1_G}$. As P has a support contained in $(\mathbb{Z}^2, 1_H)$ then $\pi(\sigma_{g^{-1}}(\hat{y})) \notin \Pi(X, f)$. Nonetheless:

$$\begin{aligned}
\sigma_{g^{-1}}(\hat{y})_{(u',1_H)} &= \hat{y}_{(u,h)(u',1_H)} \\
&= \hat{y}_{(u+\varphi_h(u'),h)} \\
&= (y^{h^{-1}})_{u'+\varphi_{h^{-1}}(u)} \\
&= (\sigma_{-\varphi_{h^{-1}}(u)}(y^{h^{-1}}))_{u'}.
\end{aligned}$$

Therefore $\pi(\sigma_{g^{-1}}(\hat{y})) = \sigma_{-\varphi_{h^{-1}}u}(y^{h^{-1}}) \in \Pi(X, f)$ which is a contradiction. Hence $\hat{y}$ does not contain any pattern from $\mathcal{F}$. It remains to show it contains no patterns in $\mathcal{G}$. Recall that patterns $P \in \mathcal{G}$ have support $\{(\vec{0}, 1_H), (\vec{0}, s^{-1})\}$ for $s \in S$. Let $g = (u, h)$ such that $\sigma_{g^{-1}}(\hat{y}) \in [P]_{1_G}$. Then $\sigma_{g^{-1}}(\hat{y})_{(\vec{0},1_H)} = (\sigma_{-\varphi_{h^{-1}}(u)}(y^{h^{-1}}))_{\vec{0}}$ and

$$\begin{aligned}
\sigma_{g^{-1}}(\hat{y})_{(\vec{0},s^{-1})} &= \hat{y}_{(u,h)(\vec{0},s^{-1})} \\
&= \hat{y}_{(u,hs^{-1})} \\
&= (y^{sh^{-1}})_{\varphi_{(sh^{-1})}u} \\
&= (\sigma_{-(\varphi_{(sh^{-1})}u)}(y^{sh^{-1}}))_{\vec{0}}.
\end{aligned}$$

Let $m \in \mathbb{N}$ and denote $(a, b) = \widetilde{\varphi}_{s^{-1}}(1, 1)$. By definition $B_m(\texttt{Sub}_{(a,b)}(y^{h^{-1}})) = (a, b)3^m + 3^{m+1}\mathbb{Z}^2$ therefore,

$$B_m(\texttt{Sub}_{(a,b)}(\sigma_{-\varphi_{h^{-1}}(u)}(y^{h^{-1}}))) = (a, b)3^m - \varphi_{h^{-1}}(u) + 3^{m+1}\mathbb{Z}^2$$

In the other hand,

$$B_m(\texttt{Sub}_{(1,1)}(\sigma_{-\varphi_{(sh^{-1})}(u)}(y^{sh^{-1}}))) = (1,1)3^m - \varphi_{(sh^{-1})}(u) + 3^{m+1}\mathbb{Z}^2.$$

So, if $\texttt{Sub}_{(a,b)}(\sigma_{g^{-1}}(\hat{y}))_{(\vec{0},1_H)} = \blacksquare$ then $\vec{0} \in (a,b)3^m - \varphi_{h^{-1}}(u) + 3^{m+1}\mathbb{Z}^2$ for some $m \in \mathbb{N}$. Applying φ_s at both sides we obtain:

$$\begin{aligned}\varphi_s(\vec{0}) = \vec{0} &\in \varphi_s(a,b)3^m - \varphi_{(sh^{-1})}(u) + 3^{m+1}\mathbb{Z}^2\\ &= \widetilde{\varphi}_s(a,b)3^m - \varphi_{(sh^{-1})}(u) + 3^{m+1}\mathbb{Z}^2\\ &= (1,1)3^m - \varphi_{(sh^{-1})}(u) + 3^{m+1}\mathbb{Z}^2\\ &= B_m(\texttt{Sub}_{(1,1)}(\sigma_{-\varphi_{(sh^{-1})}(u)}(y^{sh^{-1}}))).\end{aligned}$$

Implying that $\texttt{Sub}_{(1,1)}(\sigma_{g^{-1}}(\hat{y}))_{(\vec{0},s^{-1})} = \blacksquare$. Moreover, if either a or b is non-zero (here we treat only the $a \neq 0$ case as the $b \neq 0$ case is analogous), then, using the previous computation we get:

$$\texttt{Layer}^H_{a,s}(\sigma_{g^{-1}}(\hat{y}))_{(\vec{0},1_H)} = f_s(f_{h^{-1}}(x))_m$$

$$\texttt{Layer}^H_{1,1_H}(\sigma_{g^{-1}}(\hat{y}))_{(\vec{0},s^{-1})} = f_{sh^{-1}}(x)_m.$$

So no patterns from $\mathcal{G}$ appear, yielding $\hat{y} \in \texttt{Final}(X,f)$. □

Proposition 3.7 concludes the proof of Theorem 3.1.

4 Consequences and remarks

In this last section we explore some consequences of our simulation theorem. The first one is in the case of expansive actions. Here we show that we can replace the subaction by the projective subdynamics and obtain the same result. The second consequence is an application of Theorem 3.1 to produce non-empty strongly aperiodic subshifts of finite type in a class of groups where this was previously unknown. We also extend a Theorem of Jeandel [14] to the existence of effectively closed free dynamical systems in general.

We close this section by remarking that the technique used to prove Theorem 3.1 is valid in an even larger class (namely, simulation in $\mathbb{Z}^d \rtimes G$) and with a discussion on the size of the extension. Indeed, in Hochman's article [12] the subaction is shown to be an almost trivial isometric extension. We dedicate the last part of this section to informally discuss the size of the factor in our construction and how a similar result could be obtained.

4.1 The simulation theorem for expansive actions

Before presenting the simulation theorem for expansive actions, we must define with more detail effectively closed subshifts in groups. A longer survey of these concepts can be found in [2]. Given a group G generated by S and a finite alphabet $\mathcal{A}$ a *pattern coding* c is a finite set of tuples $c = (w_i, a_i)_{i \in I}$ where $w_i \in S^*$ and $a_i \in \mathcal{A}$. A set of pattern codings $\mathcal{C}$ is said to be recursively enumerable if there is a Turing machine which takes as input a pattern coding c and accepts it if and only if $c \in \mathcal{C}$. A subshift $X \subset \mathcal{A}^G$ is *effectively closed* if there is a recursively enumerable set of pattern codings $\mathcal{C}$ such that:

$$X = X_{\mathcal{C}} := \bigcap_{g \in G, c \in \mathcal{C}} \left(\mathcal{A}^G \setminus \bigcap_{(w,a) \in c} [a]_{gw} \right).$$

With this formal concept in hand, we show that the symbolic factors of an effectively closed G-dynamical system are still effectively closed.

Proposition 4.1. *For every finitely generated group, any G-subshift which is the factor of an effectively closed G-dynamical system is itself effectively closed.*

Proof. Let G be generated by the finite symmetric set $S \subset G$, (X, f) an effectively closed G-dynamical system over a Cantor set, (Y, σ) a G-subshift and $\phi : (X, f) \twoheadrightarrow (Y, \sigma)$ a factor.

Recall that $X \subset \{0,1\}^{\mathbb{N}}$ and $Y \subset \mathcal{A}^G$ for some finite $\mathcal{A}$. As both X and Y are compact, ϕ is uniformly continuous. Therefore for each $a \in \mathcal{A}$ then $\phi^{-1}([a]) = W_a$ where W_a is a clopen set depending on a finite number of coordinates. For any pattern coding c and $v \in S^*$:

$$\phi^{-1}\left(\bigcap_{(w,a) \in c} [a]_{vw} \right) = \bigcap_{(w,a) \in c} \phi^{-1}(\sigma_{vw}([a])) = \bigcap_{(w,a) \in c} f_{vw}(\phi^{-1}([a]))$$

Therefore,

$$Y \cap \bigcap_{(w,a) \in c} [a]_{vw} = \emptyset \implies X \cap \bigcap_{(w,a) \in c} f_{vw}(W_a) = \emptyset.$$

As (X, f) is effectively closed, there is a Turing machine which can approximate the set $\bigcap_{(w,a) \in c} f_{vw}(W_a)$ as each W_a is just a finite union of a finite intersection of cylinders and $vw \in S^*$. Also, for each partial approximation we can enumerate the cylinders which approximate the complement of X to recognize if the intersection is empty, namely, to check if $f_{vw}(W_a)$ is contained in the complement. Using these tools we can construct a Turing machine recognizing a maximal set of forbidden pattern codings defining Y. □

Theorem 4.2. *Let H be a finitely generated group and (X, f) an effectively closed expansive H-dynamical system over a Cantor set. Then there exists a $(\mathbb{Z}^2 \rtimes H)$-sofic subshift Y such that its H-projective subdynamics $\pi_H(Y)$ is conjugated to (X, f).*

Proof. Consider first (X, f) an effectively closed expansive H-dynamical system over a Cantor set. By Theorem 3.1 there exists an $(\mathbb{Z}^2 \rtimes H)$-SFT $\hat{X}$ such that its H-subaction $(\hat{X}, \sigma^H)$ is an extension of (X, f). Denote the factor map by $\phi : (\hat{X}, \sigma^H) \twoheadrightarrow (X, f)$. Let $C > 0$ be the expansivity constant of (X, f). As X is a Cantor set one can choose a clopen partition $\mathcal{P} = \{P_1, \dots, P_n\}$ such that every $P_i \in \mathcal{P}$ satisfies $\operatorname{diam}(P_i) < C$. Given $x \neq y \in X$ the expansivity implies the existence of $h \in H$ such that $d(f_h(x), f_h(y)) \geq C$. Therefore the refinement $f_h(\mathcal{P}) \vee \mathcal{P}$ separates x and y. This means that $\mathcal{P}$ is a generating partition.

Let $X_i = \phi^{-1}(P_i)$ and the continuous shift-commuting map $\hat{\phi} : \hat{X} \to \{1, \dots, n\}^{\mathbb{Z}^2 \rtimes H}$ where $\hat{\phi}(\hat{x})_{(u,h)} = i \iff \sigma_{(u,h)}(\hat{x}) \in X_i$. By definition $Y := \hat{\phi}(\hat{X})$ is a sofic $(\mathbb{Z}^2 \rtimes H)$-subshift. We claim its projective subdynamics $(\pi_H(Y), \sigma)$ are conjugate to (X, f). To see this define $\widetilde{\phi} : X \to \{1, \dots, n\}^H$ such that $\widetilde{\phi}(x)_h = i \iff x \in f_h(P_i)$. Obviously $\widetilde{\phi}$ is continuous and as $\mathcal{P}$ is generating, we have that $\widetilde{\phi}$ is injective. It is also clear by definition that $\widetilde{\phi}(X) = \pi_H(Y)$ and that $\widetilde{\phi} \circ f_h = \sigma_h \circ \widetilde{\phi}$. Therefore (X, f) is conjugate to $(\pi_H(Y), \sigma)$. □

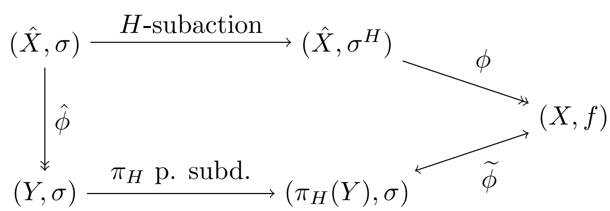


Figure 2: The diagram for the proof of Theorem 4.2.

Theorem 4.3. *Let H be a finitely generated and recursively presented group. For every effectively closed H-subshift Z there exists a sofic $(\mathbb{Z}^2 \rtimes H)$-subshift Y such that its H-projective subdynamics $\pi_H(Y)$ is Z.*

Proof. Let $S \subset H$ is a finite set such that $\langle S \rangle = H$. Consider a recursive bijection $\varphi : \mathbb{N} \to S^*$ where S^* is the set of all words on S. As H is recursively presented, then its word problem $\texttt{WP}(H) = \{w \in S^* \mid w = 1_H\}$ is recursively enumerable and there is a Turing machine T which accepts a pair $(n, n') \in \mathbb{N}^2$ if and only if $\varphi(n) = \varphi(n')$ as elements of H.

For simplicity, we suppose $Z \subset \{0,1\}^G$. Consider the map $\rho : Z \to \{0,1\}^{\mathbb{N}}$ where $\rho(z)_n = z_{\varphi(n)}$ where $\varphi(n) \in S^*$ is identified as an element of H. Consider the set $\Omega = \rho(Z)$ and the H-action $f : H \times \Omega \to \Omega$ defined as $f_h(\rho(z)) = \rho(\sigma_h(z))$. Clearly ρ is a conjugacy between (Z, σ) and (Ω, f). We claim that (Ω, f) is an effectively closed H-dynamical system.

Indeed, let $w \in \{0,1\}^*$. A Turing machine which accepts w if and only if $[w] \in \{0,1\}^{\mathbb{N}} \setminus \Omega$ is given by the following scheme: for each pair (n, n') in the support of w run T in parallel. if T accepts for a pair such that $w_n \neq w_{n'}$ then accept w (this means that w did not codify a configuration in $\mathcal{A}^{\mathbb{Z}}$ as two words codifying different group elements have different symbols). Also, in parallel, use the algorithm recognizing a maximal set of forbidden patterns for Z over the pattern coding $c_w = (\varphi(n), w_n)_{n \leq |w|}$. This eliminates all w which codify configurations containing forbidden patterns in Z. For $f_s^{-1}[w]$ just note that the application $n \to \varphi(s^{-1}\varphi^{-1}(n))$ is recursive, thus $f_s^{-1}[w]$ can be calculated.

It suffices to apply Theorem 4.2 to (Ω, f) to obtain a sofic $(\mathbb{Z}^2 \rtimes H)$-subshift Y such that $(\pi_H(Y), \sigma)$ is conjugate to (Z, σ) One can then extend this conjugacy to act over Y in such a way to obtain a factor $\hat{Y}$ of Y such that $\pi_H(\hat{Y}) = Z$. $\square$

In the case of a bigger alphabet $\mathcal{A}$, we can code each $a \in \mathcal{A}$ as a word in $\{0,1\}^k$ and redefine ρ such that for $z \in Z$ then $\rho(z)_n = (z_{\varphi(\lfloor n/k \rfloor)})_{n \mod k}$. This construction also defines a conjugated system (Ω, f) which is effectively closed.

We can describe this symbolic factor map in a simple way. Consider first the case where the alphabet is $\{0,1\}$. An explicit way to describe it is to force the recursive bijection φ described above to satisfy $\varphi(0)$ to be the empty word coding 1_H and notice that in the sofic subshift $\texttt{Final}(\Omega, f')$ the symbol z_{1_H} is therefore coded in the lattice containing x_0 in each $\mathbb{Z}^2$-coset. It suffices to use a big enough factor to recognize the first lattice in a Toeplitz layer and project to the value x_0 everywhere. In the case of a finite alphabet which is coded as words in $\{0,1\}^k$ it suffices to recognize the first k lattices and project the symbol they code.

4.2 Existence of strongly aperiodic SFT in a class of groups obtained by semidirect products

Next we show how these previous theorems can be applied to produce strongly aperiodic subshifts of finite type. We say a G-subshift (X, σ) is *strongly aperiodic* if the shift action is free, that is, $\forall x \in X, \sigma_g(x) = x \implies g = 1_G$.

Theorem 4.4. *Let H be a finitely generated group and (X, f) a non-empty effectively closed H-dynamical system which is free. Then $G \cong \mathbb{Z}^2 \rtimes H$ admits a non-empty strongly aperiodic SFT.*

Proof. We begin by recalling the following general property of factor maps. Suppose there is a factor $\phi : (X, f) \twoheadrightarrow (Y, f')$. and let $x \in X$ such that $f_g(x) = x$. Then $f'_g(\phi(x)) = \phi(f_g(x)) = \phi(x) \in Y$. This means that if f' is a free action then f is also a free action.

By Theorem 3.1 we can construct the $(\mathbb{Z}^2 \rtimes H)$-SFT $\widehat{\texttt{Final}}(X, f)$ such that $(\widehat{\texttt{Final}}(X, f), \sigma_H)$ is an extension of (X, f) via the factor $\phi_1 = \Upsilon \circ \pi \circ \phi$. We also consider the factor $\phi_2 = \texttt{Sub}_{(1,1)} \circ \phi$ which sends $\widehat{\texttt{Final}}(X, f)$ first to $\texttt{Final}(X, f)$ and then to its $\texttt{Sub}_{(1,1)}$ layer.

Let $y \in \widehat{\texttt{Final}}(X, f)$ and $(z, h) \in \mathbb{Z}^2 \rtimes H$ such that $\sigma_{(z,h)}(y) = y$. This implies that $\phi_2(y) = \sigma_{(z,h)}(\phi_2(y)) = \sigma_{(z,1_H)}(\sigma_{(0,h)}(\phi_2(y)))$. As we have seen in the proof of Theorem 3.1, the action $\sigma_{(0,h)}$ leaves the lattices $(B_m)_{m \in \mathbb{N}}$ of $\texttt{Sub}_{(1,1)}$ invariant in the $(\mathbb{Z}^2, 1_H)$-coset. Let $M > ||z||^2$. Then $\sigma_{(z,0)}$ does not leave invariant the lattice B_M. This implies that $z = \vec{0}$. Therefore, $\sigma_{(\vec{0},h)}(y) = y$. Applying ϕ_1

we obtain that $f_h(y) = y$, and thus $h = 1_H$. Therefore $(z, h) = (\vec{0}, 1_H)$ and $\widehat{\texttt{Final}}(X, f)$ is strongly aperiodic. It is non-empty as $X \neq \emptyset$. □

Theorem 4.4 requires the existence of a non-empty effectively closed free dynamical system to conclude the existence of a non-empty strongly aperiodic SFT. It is a non-trivial fact that these objects always exist when the word problem of the group is decidable. Furthermore, in the class of recursively presented groups, non-empty effectively closed subshifts which are strongly aperiodic exist if and only if the word problem of the group is decidable. This is proven in [4] and can be formally stated as follows:

Lemma 4.5. *Let H be a recursively presented group. There exists a non-empty, effectively closed and strongly aperiodic subshift over H if and only if the word problem of H is decidable.*

The only if part of this proof is a theorem by Jeandel [14] and is basically the fact that a strongly aperiodic subshift in a recursively presented group gives enough information to recursively enumerate the complement of the word problem of the group. Conversely, the existence part of the proof of Lemma 4.5 relies on a proof by Alon, Grytczuk, Haluszczak and Riordan [1] which uses Lovász local lemma to show that every finite regular graph of degree Δ can be vertex-colored with at most $(2e^{16} + 1)\Delta^2$ colors in a way such that the sequence of colors in any non-intersecting path does not contain a square word. Using compactness arguments this result is extended to Cayley graphs $\Gamma(H, S)$ of finitely generated groups where the bound takes the form $2^{19}|S|^2$ colors where $|S|$ is the cardinality of a set of generators of H. One can also show that the set of square-free vertex-coloring of $\Gamma(H, S)$ yields a strongly aperiodic subshift, which is thus non-empty if the alphabet has at least $2^{19}|S|^2$ symbols. In the case where H has decidable word problem, a Turing machine can construct the sequence of balls $B(1_H, n)$ of the Cayley graph and enumerate a codification of all patterns containing a square colored path. Therefore we obtain an effectively closed, strongly aperiodic and non-empty subshift. Using the fact that H is recursively presented one can apply the coding of Theorem 4.3 to the object of Lemma 4.5 to obtain a free non-empty effectively closed H-dynamical system (Ω, f'). Applying Theorem 4.4 to this result allows us to state the following corollary.

Corollary 4.6. *Let H be a finitely generated group with decidable word problem, then $\mathbb{Z}^2 \rtimes H$ admits a non-empty strongly aperiodic SFT.*

We remark that this corollary is an alternative proof to a construction done by Ugarcovici, Sahin and Schraudner [21] showing that the discrete Heisenberg group $\mathcal{H}$ admits non-empty strongly aperiodic SFTs. This falls directly from our theorem as $\mathcal{H} \cong \mathbb{Z}^2 \rtimes_\varphi \mathbb{Z}$ for $\varphi(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. In their proof they use a similar trick using as a base the Robinson tiling [20]. They use the lattices of crosses in this object to match the different $(\mathbb{Z}^2, 0)$-cosets correctly to force a trivial action in the $\mathbb{Z}$ direction and use a counter machine to create aperiodicity in the other direction. In our construction the Robinson tiling got replaced by the substitutive subshifts $\texttt{Sub}_{(a,b)}$ which are able to match correctly the cosets of any possible automorphism and the counter machine by the simulation of the free H-dynamical system. We also remark that Corollary 4.6 answers some open questions in their talk asking the same property for the Flip, Sol groups and the powers of the Heisenberg group which can be represented as $\mathbb{Z}^2 \rtimes_\varphi \mathbb{Z}$ for φ given by the matrices $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, $\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}$ and $\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}$ respectively. The only case in their list which is not solved is a two-dimensional Baumslag Solitar group which we don't know how to express as a semidirect product.

A theorem of Jeandel [14] says that for recursively presented groups G, the existence of a non-empty strongly aperiodic subshift $X \subset \mathcal{A}^G$ implies that the word problem of G is decidable. We can extend this to the case of arbitrary dynamical systems. This gives a deep relation between computability and dynamical properties.

Corollary 4.7. *Let H be a recursively presented and finitely generated group. There exists a strongly aperiodic effectively closed H-dynamical system if and only if the word problem of H is decidable.*

Proof. If the word problem of H is decidable, we can use the effectively closed subshift constructed in [4] as an example. Conversely, Jeandel's result implies that if a recursively presented group admits a non-empty effectively closed and strongly aperiodic subshift then it's word problem is decidable. Using Theorem 4.4 we can construct a strongly aperiodic subshift from any free effectively closed H-dynamical system. Therefore the word problem of H is decidable. □

4.3 A generalization and comments on the size of the extension

In this last portion we want to make explicit that the technique used in the proof of Theorem 3.1 can be easily be generalized to the following context

Theorem 4.8. *Let H be finitely generated group, $d \geq 2$ and $G = \mathbb{Z}^d \rtimes H$. For every H-effectively closed dynamical system (X, f) there exists a G-SFT whose H-subaction is an extension of (X, f).*

Indeed, instead of considering vectors in $(\mathbb{Z}/3\mathbb{Z})^2 \setminus \{\vec{0}\}$ we use $v \in (\mathbb{Z}/3\mathbb{Z})^d \setminus \{\vec{0}\}$ and d-dimensional substitutions $\texttt{s}_v$ defined analogously. The subshifts generated by these substitutions carry $\mathbb{Z}^d$-lattices and the configurations $z \in \texttt{Sub}_v$ can be described in the same way as before by lattices $B_m(z)$. The Toeplitz construction $\texttt{Top}(X, f)$ stays the same but instead of just constructing $\texttt{Top}(X, f)^H$ and $\texttt{Top}(X, f)^V$ we construct $\texttt{Top}(X, f)^{e_i}$ for every canonical vector $\{e_i\}_{i \leq d}$ where the $\langle e_i \rangle$-projective subdynamics yields $\texttt{Top}(X, f)$ and the configurations are extended periodically everywhere else. The rest of the construction translates directly to this setting and Theorem 4.4 and Corollary 4.6 also hold.

We also want to remark the following: Hochman's theorem gives further information about the extension. Formally, given an effectively closed $\mathbb{Z}^d$-dynamical system. A $\mathbb{Z}^{d+2}$-SFT is constructed such that the $\mathbb{Z}^d$-subaction is an *almost trivial isometric extension* (ATIE) of the $\mathbb{Z}^d$-dynamical system. An extension $(Z, f_Z) \twoheadrightarrow (Y, f_Y)$ is an ATIE if we can interpolate a factor

$$(Z, f_Z) \twoheadrightarrow (Y, f_Y) \times (W, f_W) \twoheadrightarrow (Y, f_Y)$$

such that (W, f_W) is an isometric action of a totally disconnected space, $(Y, f_Y) \times (W, f_W) \twoheadrightarrow (Y, f_Y)$ is the projection of the first coordinate $(Z, f_Z) \twoheadrightarrow (Y, f_Y) \times (W, f_W)$ is almost everywhere $1-1$, that is, it satisfies that the set of points with unique preimage has measure 1 under any invariant Borel probability measure.

The idea behind the notion of ATIE is of an extension which is in a certain sense "small". It consists basically on adding a simple system (W, f_W) as a product and then considering a measure equivalent action as the extension. Many properties such as the topological entropy (at least for $\mathbb{Z}^d$-actions) are preserved by taking ATIEs.

In our construction the only obstruction towards obtaining an ATIE is the use of the simulation theorem of effectively closed $\mathbb{Z}$-subshifts as projective subactions of sofic $\mathbb{Z}^2$-subshifts. This theorem in its current state does not yield an almost everywhere $1-1$ extension. The rest of the proof can be adapted to obtain an ATIE, for instance, the substitutive layers can be coupled in a single substitution to avoid the degree of freedom when either a or b are zero. Furthermore, the substitutive layers and the Toeplitz structure can be factorized in the isometric action as they are invariant under the H-subaction. Therefore, the maps $\Upsilon \circ \pi$ do not pose obstructions to obtaining an ATIE. Everything that remains is the factor $\phi : \widehat{\texttt{Final}}(X, f) \twoheadrightarrow \texttt{Final}(X, f)$. Here the substitutive layers don't present a problem as they come from a primitive substitution with unique derivation and thus Mozes's theorem [18] gives the almost $1-1$ SFT extension. The only thing that remains is the aforementioned almost $1-1$ SFT extension for $\texttt{Top}(X, f)^H$ and $\texttt{Top}(X, f)^V$ that could be obtained by refining that simulation theorem.

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