The domino problem for self-similar structures

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Abstract. We define the domino problem for tilings over self-similar structures of $\mathbb{Z}^d$ given by forbidden patterns. In this setting we exhibit non-trivial families of subsets with decidable and undecidable domino problem.

Introduction

In its original form, the domino problem was introduced by Wang [10] in 1961. It consists of deciding if copies of a finite set of Wang's tiles (square tiles of equal size, not subject to rotation and with colored edges) can tile the plane subject to the condition that two adjacent tiles possess the same color in the edge they share. Wang's student Berger showed undecidability for the domino problem on the plane in 1964 [3] by using a reduction to the halting problem. In 1971, Robinson [8] simplified Berger's proof.

Symbolic dynamics classically studies sets of colorings of $\mathbb{Z}^d$ from a finite set of colors which are closed in the product topology and invariant by translation, such sets are called subshifts. Given a finite set of patterns $\mathcal{F}$ (a pattern is a coloring of a finite part of $\mathbb{Z}^d$), we associate a subshift of finite type $X(\mathcal{F})$ which corresponds to the set of colorings which does not contain any occurrence of patterns in $\mathcal{F}$. The domino problem can therefore be expressed in this setting: given a finite set of forbidden patterns $\mathcal{F}$, is it possible to decide whether the subshift of finite type $X(\mathcal{F})$ is not empty?

It is well known that there exists an algorithm deciding if a subshift of finite type is empty in dimension one [5] and that there is no such algorithm in higher dimensions. The natural question that comes next is: *What is the frontier between decidability and undecidability in the domino problem?*

One way to explore this question is to consider subshifts defined over more general structures, such as finitely generated groups or monoids and ask where the domino problem is decidable. This approach has yielded various result in different structures: Some examples are the hyperbolic plane [6], confirming a conjecture of Robinson [9] and Baumslag-Solitar groups [1]. The conjecture in this direction is that the domino problem is decidable if and only if the group is virtually free. The conjecture is known to hold in the case of virtually nilpotent groups [2]. The main idea of the proof of this result is to construct a grid by local rules in order to use the classical result in $\mathbb{Z}^2$.

In this paper we explore another way to delimit the frontier between decidability and undecidability of this problem. In geometry the structures which lie between the line and the plane can have Hausdorff dimension strictly between one and two. In this article we propose a way to define the domino problem in a digitalization of such fractal structures. In Section 1 we use self-similar substitutions to define a "fractal" structure where a natural version of the domino problem can be defined. We exhibit a large class of substitutions (including the one which represents the Sierpiński triangle) where the domino problem is decidable (Section 2), another class (including the Sierpiński carpet) where the problem is undecidable (Section 4) and an intermediate class where the question is still open (see Figure 1 for an example of each of these classes).

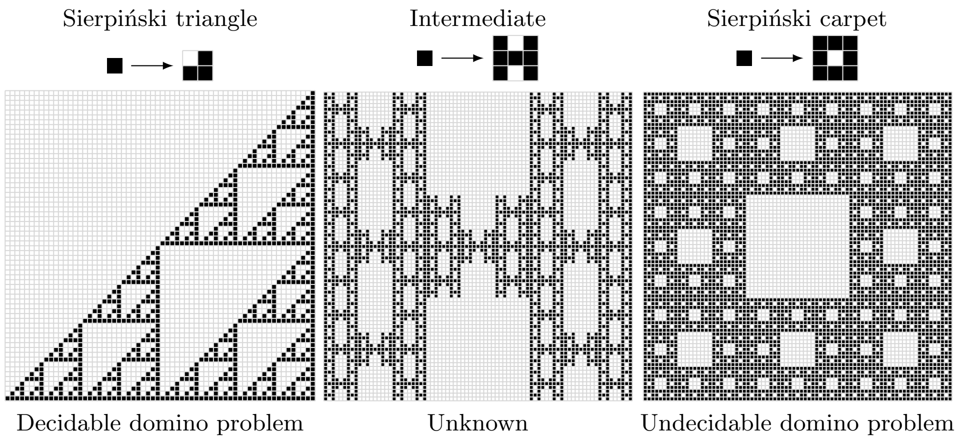


Fig. 1. Some digitalizations of fractal structures and the status of their domino problem

1 Position of the problem

1.1 Coloring of $\mathbb{Z}^d$ and local rules

Given a finite alphabet $\mathcal{A}$, a coloring of $\mathbb{Z}^d$ is called a *configuration*. The set of configurations, denoted $\mathcal{A}^{\mathbb{Z}^d}$, is a compact set according to the usual product topology. A *subshift* is a closed set of configurations which is invariant by the shift action. Given a finite subset $S \subset \mathbb{Z}^d$, a *pattern with support* S is an element p of $\mathcal{A}^S$. A pattern $p \in \mathcal{A}^S$ *appears* in a configuration $x \in \mathcal{A}^{\mathbb{Z}^d}$ if there exists $z \in \mathbb{Z}^d$ such that $x_{z+S} = p$. In this case we write $p \sqsubset x$.

Equivalently, a subshift can be defined with a set of forbidden patterns $\mathcal{F}$ as the set of configurations where no patterns of $\mathcal{F}$ appear. We denote it by $X(\mathcal{F})$. If $\mathcal{F}$ is finite, $X(\mathcal{F})$ is called *subshift of finite type* which can be considered as the set of tilings defined by the local contraints given by $\mathcal{F}$.

1.2 Self similar structures

We want to extend the condition of coloring to self-similar structures of $\mathbb{Z}^d$. This means that only some cells can be decorated by elements of $\mathcal{A}$. To formalize that, a structure is coded as a subset of $\{0,1\}^{\mathbb{Z}^d}$ and self-similarity is obtained by a substitution.

Let $\mathcal{A}$ be a finite alphabet. A *substitution* is a function $s : \mathcal{A} \to \mathcal{A}^R$ where $R = [1, l_1] \times \cdots \times [1, l_d]$ is a d-dimensional rectangle. It is naturally extended to act over patterns which have rectangles as support by concatenation. We denote the successive iterations of s over a symbol by s, s^2, s^3 and so on. The subshift generated by a substitution s is the set

$$X_s := \{x \in \mathcal{A}^{\mathbb{Z}^d} | \forall p \sqsubset x, \exists n \in \mathbb{N}, a \in \mathcal{A}, p \sqsubset s^n(a)\}.$$

To obtain self-similar structures, we restrict the notion of substitution to $\{0,1\}$ imposing that the image of 0 consists of a block of 0s. These substitutions are called *self-similar*. Self-similar substitutions represent digitalizations of the iterations of the following procedure: start with the hypercube $[0,1]^d$, subdivide it in a $l_1 \times \cdots \times l_d$ grid and remove the blocks in the positions z of the grid where $s(1)_z = 0$. Then repeat the same procedure with every sub-block.

Example 1. Consider $\mathcal{A} = \{□, ■\}$. The self-similar substitution s such that:

is called the Sierpiński triangle substitution and is extended by concatenation as shown in Figure 2.

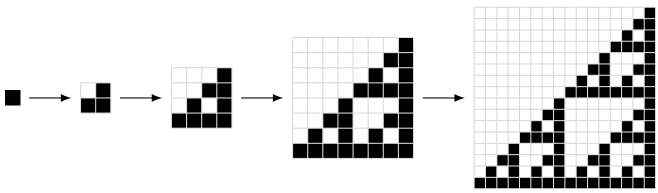


Fig. 2. First four iterations of the Sierpiński triangle substitution.

1.3 Coloring of a self-similar structure and local rules

Let $\mathcal{A}$ be a finite alphabet where $0 \in \mathcal{A}$ and s be a self-similar substitution. Consider $X_s \subset \{0,1\}^{\mathbb{Z}^d}$ the associated self-similar structure. A configuration $x \in \mathcal{A}^{\mathbb{Z}^d}$ is *compatible* with s if $\pi(x) \in X_s$ where π is a map which sends all elements of $\mathcal{A} \setminus \{0\}$ onto 1 and 0 onto 0. Given a finite set of patterns $\mathcal{F}$ we define the *set of configurations on X_s defined by local rules $\mathcal{F}$* as

$$X_s(\mathcal{F}) = \left\{ x \in \mathcal{A}^{\mathbb{Z}^d} : \pi(x) \in X_s \text{ and no pattern of } \mathcal{F} \text{ appears in } x \right\}.$$

1.4 The domino problem on self-similar structures

The domino problem for $\mathbb{Z}^d$ is defined as the language

$$\mathtt{DP}(\mathbb{Z}^d) = \{\mathcal{F} \text{ finite set of patterns} : X(\mathcal{F}) \neq \emptyset\}.$$

It is the language of all finite sets of patterns over a finite alphabet such that it is possible to construct a configuration without patterns of $\mathcal{F}$.

Classical results which can be found in [5] show that the domino problem for $\mathbb{Z}$ is decidable. In the other hand, we know that for $d > 1$ the domino problem for $G = \mathbb{Z}^d$ is undecidable (see [3, 8]). This gap of decidability when the dimension increases motivates us to define the domino problem for structures which lay between those groups. Thus given a self-similar substitution s we introduce the *s-based domino problem* as the language

$$\mathtt{DP}(s) := \{\mathcal{F} \text{ finite set of patterns} : X_s(\mathcal{F}) \neq \{0^{\mathbb{Z}^d}\}\}.$$

That is, $\mathtt{DP}(s)$ is the set of all finite sets of forbidden patterns such that there is at least a configuration containing a non-zero symbol. We assume implicitly that $\mathcal{F}$ does not contain any pattern consisting only of 0s. By a compactness argument, $\mathtt{DP}(s)$ can be equivalently defined as the set of $\mathcal{F}$ such that for arbitrarily big $n \in \mathbb{N}$ the non-zero symbols in $s^n(1)$ can be colored avoiding all patterns in $\mathcal{F}$.

2 Self-similar structures with decidable domino problem

In this section we present a family of self-similar substitutions such that their domino problem is decidable. In order to present this result in the most general setting, we introduce the channel number of a self-similar substitution.

Let $\mathbb{H} = \{-1, 0, 1\}^d$ and consider the set $\Lambda \subset \{0, 1\}^{\{1,2,3\}^d}$ consisting of all d-dimensional hypercube patterns of side 3 which appear in X_s and that have a 1 in the center. Let $\Lambda_n = s^n(\Lambda)$ be the set of the images of each $q \in \Lambda$ under s^n by concatenation and S_n be the support corresponding to the image of position $(2, \dots, 2)$ of q under s^n. We define the *n-channel number* $\chi(s, n)$ of s as follows:

$$\chi(s, n) = \max_{p \in \Lambda_n} |\{z \in S_n \mid \exists h \in (z + \mathbb{H}) \cap (supp(p) \setminus S_n), p_z = p_h = 1\}|$$

In other words, it is the maximum number of positions in the support of the pattern $s^n(1)$ such that if we surround it either by blocks of 0 or copies of $s^n(1)$ appearing in X_s there might be two symbols 1, one appearing in $s^n(1)$ and another outside, at distance smaller than 1. We say that s is *channel bounded* if there exists $K \in \mathbb{N}$ such that for all n, $\chi(s, n) \leq K$. The Sierpiński triangle substitution from Figure 2 is an example of a channel bounded substitution as colorings of $s^n(1)$ can be constructed by pasting three colorings of $s^{n-1}(1)$ and forbidden patterns can only appear locally around the corners.

Theorem 1. *For every channel bounded self-similar substitution s the domino problem $\mathtt{DP}(s)$ is decidable.*

Proof. Let $\mathcal{F}$ be a set of forbidden patterns over an alphabet $\mathcal{A}$. It suffices to show that if s is channel bounded, it is possible to calculate $N \in \mathbb{N}$ with the property that the existence of any coloring of $s^N(1)$ with symbols from $\mathcal{A}$ without any subpattern from $\mathcal{F}$ implies $X_s(\mathcal{F}) \neq \{0^{\mathbb{Z}^d}\}$. Indeed, an algorithm could calculate N and try every coloring of $s^N(1)$. If there exists one where no pattern in $\mathcal{F}$ appears it returns that $X_s(\mathcal{F}) \neq \{0^{\mathbb{Z}^d}\}$, otherwise it returns $X_s(\mathcal{F}) = \{0^{\mathbb{Z}^d}\}$.

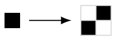
For simplicity, suppose that $\forall p \in \mathcal{F}, supp(p) \subseteq \mathbb{H}$ and let K be a bound for $\chi(s,n)$ (If the support is $\{-m,\dots,m\}^d$ we can recalculate a new K). We claim that $N := 2^{|\mathcal{A}|^{(3^d-1)K}}$ suffices. For each $q \in \Lambda$ consider the a coding $J_q = \{j_1,\dots,j_{k_q}\}$ with $k_q \leq K$ of the positions from the definition of $\chi(s,n)$. That is, J_q codes for all $n \in \mathbb{N}$ the set of positions which matter when considering only q. Any recursive ordering similar to the one given by a Quadtree works. Consider a coloring of $s^n(1)$ without subpatterns in $\mathcal{F}$ and store the symbols of this coloring appearing in J_q as a tuple $(a_{j_1}, a_{j_2}, \dots a_{j_{k_q}}) \in \mathcal{A}^{J_q}$. Therefore all the information concerning the dependency of this coloring with its possible surroundings can be stored on $|\Lambda|$ tuples. Now, given the set of all colorings of $s^n(1)$ which do not contain any forbidden pattern we can extract the $|\Lambda|$ tuples from each one of them. All this information for the level n is represented as a subset of $\prod_{q\in\Lambda} \mathcal{A}^{J_q}$. By definition this is the only information needed in order to make sure of the existence of a coloring of $s^{n+1}(1)$ with no subpatterns in $\mathcal{F}$. Moreover, the tuples representing those patterns can be obtained from the ones of $s^n(1)$ because the positions from the definition of $\chi(s,n+1)$ necessarily appear in the patterns of $s^n(1)$. This means it is possible to extract pasting rules which can be codified in a function $\mu_s : 2^{\prod_{q\in\Lambda} \mathcal{A}^{J_q}} \to 2^{\prod_{q\in\Lambda} \mathcal{A}^{J_q}}$.

This function codes how to construct the tuples of level $n+1$ from the tuples of level n. Obviously, $\mu_s(\emptyset) = \emptyset$, therefore there are two possibilities: either this function arrives eventually at $\emptyset$ and there are no colorings of $s^m(1)$ for some $m \in \mathbb{N}$ or μ_s cycles and thus it's possible to construct colorings of $s^m(1)$ for arbitrarily big m. By pigeonhole principle this behavior must occur before $|2^{(\mathcal{A}^K)^\Lambda}| \leq 2^{|\mathcal{A}|^{(3^d-1)K}}$ iterations of μ_s.

3 The Mozes property for self-similar structures

Most of the proofs of the undecidability of the domino problem on $\mathbb{Z}^2$ are based on the construction of a self-similar structure. A Theorem proven by Mozes [7] shows that every $\mathbb{Z}^d$-substitutive subshift is a sofic subshift for $d \geq 2$. This theorem fails for the case $d = 1$. The importance of this result is the fact that multidimensional substitutions can be realized by local rules. In order to present a family of self-similar substitutions with undecidable domino problem we will make use of an analogue of the theorem shown by Mozes.

Definition 1. *A self-similar substitution s satisfies the Mozes property if for every substitution s' defined over the same rectangle and over an alphabet $\mathcal{A}$ containing 0 and such that $\forall a \in \mathcal{A} \setminus \{0\}$, $\pi(s'(a)) = s(1)$ and $s'(0) = s(0)$ there* *exists an alphabet $\mathcal{B}$ containing the symbol* 0*, a finite set of forbidden patterns* $\mathcal{F} \subseteq \mathcal{B}^*_{\mathbb{Z}^d}$ *and a local function* $\Phi : \mathcal{B} \to \mathcal{A}$ *such that* $\Phi(0) = 0$ *and the function* $\phi : \mathcal{B}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$ *given by* $\phi(x)_z = \Phi(x_z)$ *is surjective from* $X_s(\mathcal{F})$ *to* $X_{s'}$*.*

In other words, it's the analogue of saying that $X_{s'}$ is a sofic subshift, except that the SFT extension has to be based on X_s. Currently, we have been able to produce a class of substitutions that satisfy the Mozes property but we have not found a characterization of those who do. An example of a substitution without the Mozes property is the one given by .

An interesting example of a substitution satisfying the Mozes property is the Sierpiński carpet shown in Figure 3. This substitution it not channel bounded as at least one of the four borders of a coloring of $s^{n-1}(1)$ matter when constructing $s^n(1)$ and thus $\chi(s, n)$ grows exponentially. In fact this substitution belongs to a bigger class which also satisfies the Mozes property. In the next section we introduce this class and use this previous fact to prove that all the substitutions belonging to it have undecidable domino problem.

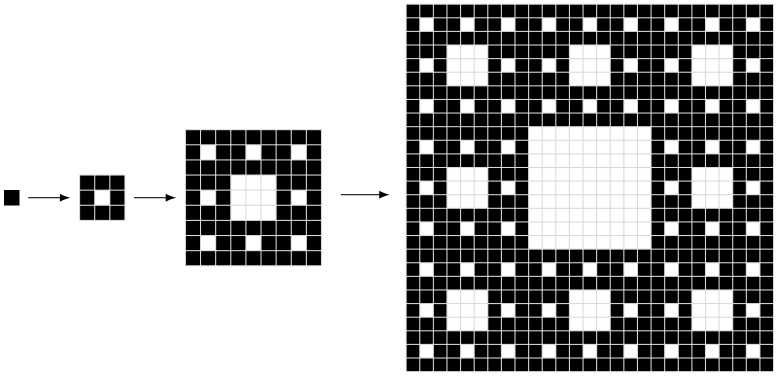


Fig. 3. The first iterations of the Sierpiński carpet substitution.

4 Self-similar structures where the domino problem is undecidable

In this section we present a family of two-dimensional self-similar substitutions with undecidable s-based domino problem. The definition of this class follows.

Definition 2. *A self-similar substitution s defined on* $[1, l_1] \times [1, l_2]$ contains a grid *if there are integers* $1 \leq i_1 < i_2 <= l_1$ *and* $1 \leq j_1 < j_2 <= l_2$ *such that* $j \in \{j_1, j_2\}$ *or* $i \in \{i_1, i_2\}$ *implies that* $s(1)_{(i,j)} = 1$*.*

One example of a self-similar substitution that contains a grid is the Sierpiński carpet. One interesting property of these substitutions is that they satisfy the Mozes property. This follows from a technical construction which uses a layer that looks like a generalized Robinson tiling [8] and stores the information of the simulated substitution and its past hierarchically.

Theorem 2. *All self-similar substitutions which contain a grid satisfy the Mozes property.*

In what remains of this section we show the following theorem:

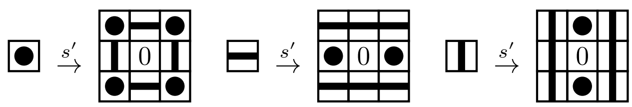
Theorem 3. *Let s be a self-similar substitution which contains a grid. Then the domino problem $\mathtt{DP}(s)$ is undecidable.*

Proof. We claim that an oracle for $\mathtt{DP}(s)$ can be used to decide $\mathtt{DP}(\mathbb{Z}^2)$. This is enough to conclude, as $\mathtt{DP}(\mathbb{Z}^2)$ is undecidable.

Let s be defined on $[1, l_1] \times [1, l_2]$, some values satisfying the grid condition (i_1, i_2) and (j_1, j_2) and consider a substitution s' over the alphabet $\mathcal{A}(s') = \{\boxdot, \boxminus, \boxbar, 0\}$ given by the following rules: Let $C = \{(i_1, j_1), (i_1, j_2), (i_2, j_1), (i_2, j_2)\}$, $H = \{(i, j) | j \in \{j_1, j_2\}\} \setminus C$ and $V = \{(i, j) | i \in \{i_1, i_2\}\} \setminus C$.

$$s'(\boxdot)_z = \begin{cases} 0, & \text{if } s(1)_z = 0 \\ \boxminus, & \text{if } z \in H \\ \boxbar, & \text{if } z \in V \\ \boxdot, & \text{else} \end{cases} \quad s'(\boxbar)_z = \begin{cases} 0, & \text{if } s(1)_z = 0 \\ \boxbar, & \text{if } z \in V \cup C \\ \boxdot, & \text{else} \end{cases} \quad s'(\boxminus)_z = \begin{cases} 0, & \text{if } s(1)_z = 0 \\ \boxminus, & \text{if } z \in H \cup C \\ \boxdot, & \text{else} \end{cases}$$

For example, in the case where s is the Sierpiński carpet we get:



For any $y \in X_{s'} \setminus \{0^{\mathbb{Z}^2}\}$ and $n \in \mathbb{N}$ we have $s'^n(\boxdot) \sqsubset y$. Indeed, $\boxdot$ appears in the image of every symbol $a \in \mathcal{A}(s') \setminus \{0\}$. This implies that for every positive integer n, a $\boxdot$ must appear at a bounded distance of every non-zero symbol in $s'^n(a)$. This argument extends inductively because if $s'^{n-1}(\boxdot)$ appears at a bounded distance in every $s'^k(a)$ with $k > n$, it suffices to apply s' to obtain that $s'^n(\boxdot)$ appears at bounded distance in $s'^{k+1}(a)$.

As s satisfies the Mozes property there exists an alphabet $\mathcal{B}(s')$, a finite set $\mathcal{F}(s') \subset \mathcal{B}(s')^*_{\mathbb{Z}^2}$ and $\Phi : \mathcal{B}(s') \to \mathcal{A}(s')$ such that $\Phi(0) = 0$ and the function $\phi : \mathcal{B}(s')^{\mathbb{Z}^d} \to \mathcal{A}(s')^{\mathbb{Z}^d}$ given by $\phi(x)_z = \Phi(x_z)$ is surjective from $X_s(\mathcal{F})$ to $X_{s'}$.

Consider a finite set of forbidden patterns $\mathcal{F}$ over an alphabet $\mathcal{A}$ defining a $\mathbb{Z}^2$ subshift $X(\mathcal{F})$. Without loss of generality $\mathcal{F}$ contains only patterns with supports $\{(0, 0), (1, 0)\}$ and $\{(0, 0), (0, 1)\}$ (one can choose a conjugated version of $X(\mathcal{F})$ satisfying this property by using a higher block code. See [5]).

Finally, consider the alphabet $\mathcal{S} := \mathcal{B}(s') \times (\mathcal{A} \cup \{0\})$ along with the set of forbidden patterns $\mathcal{G}$ given by the union of the following sets:

- Zeros correspond: $\{(0,a) \mid a \in \mathcal{A}\} \cup \{(b,0) \mid b \in \mathcal{B}(s') \setminus \{0\}\}$.
- First layer forbidden patterns: $\{p \times q \mid p \in \mathcal{F}(s'), q \in \mathcal{A}^{supp(p)}\}$. These forbidden patterns make sure that configurations belonging to the first layer of $X_s(\mathcal{G})$ belong to $X_s(\mathcal{F}(s'))$.
- Horizontal forbidden patterns: let $p \in \mathcal{S}^{\{(0,0),(1,0)\}}$ be denoted by (a,b,c,d) if $p(0,0) = (a,c)$ and $p(1,0) = (b,d)$ and $q \in \mathcal{A}^{\{(0,0),(1,0)\}}$ be denoted by (c,d) if $q(0,0) = c$ and $q(1,0) = d$. The set of horizontal forbidden patterns is $\{(a,b,c,d) \mid (a = \boxminus, b \in \{\boxminus, \boxdot\}$ and $c \neq d)$ or $(a = \boxdot, b = \boxminus$ and $(c,d) \in \mathcal{F})\}$.
- Vertical forbidden patterns: let $p \in \mathcal{S}^{\{(0,0),(0,1)\}}$ be denoted by (a,b,c,d) if $p(0,0) = (a,c)$ and $p(0,1) = (b,d)$ and $q \in \mathcal{A}^{\{(0,0),(0,1)\}}$ be denoted by (c,d) if $q(0,0) = c$ and $q(0,1) = d$. The set of vertical forbidden patterns is given by $\{(a,b,c,d) \mid (a = ◫, b \in \{◫, \boxdot\}$ and $c \neq d)$ or $(a = \boxdot, b = ◫$ and $(c,d) \in \mathcal{F})\}$.

These rules codify the following idea: $\boxdot$s carry arbitrary symbols from $\mathcal{A}$ in the second layer and the arrows send this information left and up respecting the rules from $\mathcal{F}$, see Figure 4. By iterating the substitution s it is easy to see that $s^n(1)$ actually contains 2^n vertical and horizontal lines. This means that the intersections of these lines contain symbols of $\mathcal{A}$ which represent a $2^n \times 2^n$ pattern which contains no forbidden pattern from $\mathcal{F}$. Therefore if $X_s(\mathcal{G}) \neq \{0^{\mathbb{Z}^2}\}$ then $X(\mathcal{F}) \neq \emptyset$ by compactness. Conversely if $X(\mathcal{F}) \neq \emptyset$ it is possible to always build the second layer of a point having $s'^n(1)$ in the first layer.

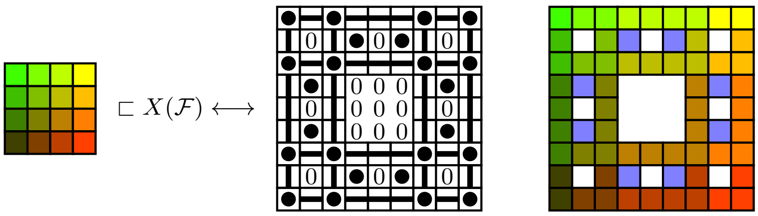


Fig. 4. On the left a pattern from $X(\mathcal{F})$. In the right its coding in $X_s(\mathcal{G})$. The blue squares are arbitrary symbols from $\mathcal{A}$.

Suppose there is an algorithm for deciding $\mathtt{DP}(s)$. Then for any $\mathcal{F}$ defining a $\mathbb{Z}^2$ subshift the alphabet $\mathcal{S}$ and the rules $\mathcal{G}$ can be built in order to decide if $X_s(\mathcal{G}) \neq \{0^{\mathbb{Z}^2}\}$. This is equivalent to deciding if $X(\mathcal{F}) \neq \emptyset$, therefore $\mathtt{DP}(\mathbb{Z}^2)$ can be decided. This yields the desired contradiction.

5 Generalizations and perspectives

Here we present some ideas to generalize previous results in order to advance towards a characterization of the self-similar structures where the domino problem is decidable. In the previous sections the information which allows to simulate grids is transferred through straight lines. We can imagine less rigid possibilities.

5.1 Connectivity

We propose a way to define the directions in which the information can be transfered in a substitution in $\mathbb{Z}^2$. Given a self-similar substitution defined over $[1, l_1] \times [1, l_2]$ we denote by $\mathbb{X}$ the set of coordinates z such that $s(1)_z = 1$. Let $\mathbb{S} = \{(0, -1), (0, 1), (-1, 0), (0, 1)\}$ and $\mathbb{W}$ contains $\{(-1, -1), (1, 1)\}$ if $\{(1, 1), (l_1, l_2)\} \in \mathbb{X}$ and $\{(-1, 1), (1, -1)\}$ if $\{(1, l_2), (l_1, 1)\} \in \mathbb{X}$. We say s admits a rigid (respectively flexible) vertical line at $1 \leq v \leq l_1$ if there is a non-repeating sequence of vertices $(v, 1) = x_1, \ldots x_n = (v, l_2)$ such that the differences $x_j - x_{j-1}$ belong to $\mathbb{S}$ (respectively $\mathbb{W} \cup \mathbb{S}$). We define rigid and flexible horizontal lines for $1 \leq h \leq l_2$ analogously. We also say that two lines are weakly disjoint if they share no consecutive pair of vertices in their path.

According to these notions, we distinguish the following four subclasses:

- s has *bounded connectivity* if s has at most one flexible horizontal and vertical line;
- s has *a isthmus* if $s(1)$ has at least two weakly disjoint flexible lines in one direction and at most one weakly disjoint flexible line in the other direction;
- s has *a weak grid* if $s(1)$ has at least two flexible horizontal lines and two flexible vertical lines which are pairwise weakly disjoint.
- s has *a strong grid* if $s(1)$ has at least two rigid horizontal lines and two rigid vertical lines which are pairwise weakly disjoint.

If s has bounded connectivity the proof of Theorem 1 can be adapted to show decidability. If s has a strong grid it is possible to adapt the proof of Theorem 3 to show the undecidability of the domino problem associated to such substitution, moreover, a generalization of that proof works even in the case of weak grids. Nevertheless we still have no results supporting either direction in the isthmus case. We believe that the Mozes property does not hold in the isthmus case, which would be evidence towards decidability. Figure 5 presents the domino problem of different substitutions according to this classification.

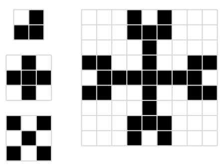
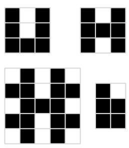
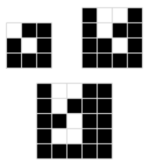
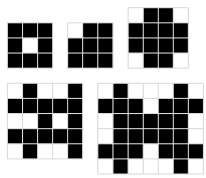
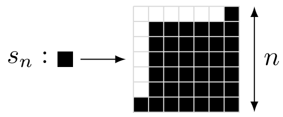
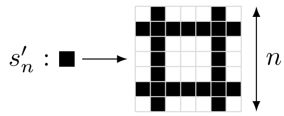
B. Connectivity	Isthmus	Weak grid	Strong grid
DP decidable	Unknown	DP undecidable	DP undecidable
			

Fig. 5. Some examples of substitution according to this classification

5.2 Concluding remarks

In this article we introduced a version of the domino problem on self-similar structures in order to understand the frontier between decidability and undecidability in the domino problem when we go from the line (dimension 1) to the plane (dimension 2). In fact it does not depend on the Hausdorff dimension of the self-similar structure considered. Indeed, using the obtained results it is possible to obtain self-similar structures with decidable domino problem and Hausdorff dimension arbitrary near to 2 (obtained by s_n) and self-similar structures with undecidable domino problem and Hausdorff dimension arbitrary near to 1 (obtained by s'_n) .





Thus, the frontier between decidability and undecidability seems more likely to be based on the presence of a grid where it is possible to implement a computation. To confirm this hypothesis, it remains to study self-similar structures with an isthmus. In the case of an isthmus the substitution presents an unique bridge which links different zones. This prevents the possibility of a Mozes-like [7] or Goodman-Strauss-like [4] proof of the Mozes property and therefore of the implementation of a computation. The main problem is that in order to simulate a substitution there is the need to transfer arbitrarily big amounts of information by that isthmus. We believe the study of this class of substitutions will certainly provide new tools to the study of how information can be transfered.

References

1. Nathalie Aubrun and Jarkko Kari. Tiling problems on Baumslag-Solitar groups. *Preprint*, 2013.
2. Alexis Ballier and Maya Stein. The domino problem on groups of polynomial growth. *arXiv preprint arXiv:1311.4222*, 2013.
3. Robert Berger. *The Undecidability of the Domino Problem.* American Mathematical Society, 1966.
4. Chaim Goodman-Strauss. Matching rules and substitution tilings. *Annals of Mathematics*, 147(1):pp. 181–223, 1998.
5. Douglas A. Lind and Brian Marcus. *An Introduction to Symbolic Dynamics and Coding.* Cambridge University Press, 1995.
6. Maurice Margenstern. The domino problem of the hyperbolic plane is undecidable. *Bulletin of the EATCS*, pages 220–237, 2007.
7. Shahar. Mozes. Tilings, substitution systems and dynamical systems generated by them. *Journal d'Analyse Mathématique*, 53(1):139–186, 1989.
8. Raphael M. Robinson. Undecidability and nonperiodicity for tilings of the plane. *Inventiones Mathematicae*, 12:177–209, 1971.
9. Raphael M. Robinson. Undecidable tiling problems in the hyperbolic plane. *Inventiones Mathematicae*, 44:159–264, 1978.
10. Hao Wang. Proving theorems by pattern recognition i. *Commun. ACM*, 3(4):220–234, April 1960.