On the entropies of subshifts of finite type on countable amenable groups

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Abstract

Let G, H be two countable amenable groups. We introduce the notion of group charts, which gives us a tool to embed an arbitrary H-subshift into a G-subshift. Using an entropy addition formula derived from this formalism we prove that whenever H is finitely presented and admits a subshift of finite type (SFT) on which H acts freely, then the set of real numbers attained as topological entropies of H-SFTs is contained in the set of topological entropies of G-SFTs modulo an arbitrarily small additive constant for any finitely generated group G which admits a translation-like action of H. In particular, we show that the set of topological entropies of G-SFTs on any such group which has decidable word problem and admits a translation-like action of $\mathbb{Z}^2$ coincides with the set of non-negative upper semi-computable real numbers. We use this result to give a complete characterization of the entropies of SFTs in several classes of groups.

Key words and phrases: topological entropy, symbolic dynamics, subshifts of finite type, amenable groups, cocycles of group actions.

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1 Introduction

The topological entropy of an action $G \curvearrowright X$ of an amenable group G on a compact metric space X by homeomorphisms is a non-negative number which counts the asymptotic exponential growth rate of the number of distinguishable orbits of the system. Initially introduced by Adler, Konheim and McAndrew [1] for $\mathbb{Z}$-actions, it is an important conjugacy invariant which has been studied broadly.

A particularly interesting case is when $G \curvearrowright X$ is a subshift of finite type (G-SFT). Up to dynamical conjugacy, there are countably many distinct subshifts of finite type, and therefore at most countably many real numbers can be attained as the entropy of a subshift of finite type. A classical result by Lind [20] classifies the topological entropies attainable by $\mathbb{Z}$-SFTs as non-negative rational multiples of logarithms of Perron numbers. This characterization relies on a full description of the configurations of $\mathbb{Z}$-SFTs as bi-infinite paths on a finite graph and a study of the eigenvalues of their adjacency matrices.

A more recent result by Hochman and Meyerovitch [14] completely classifies the entropies of $\mathbb{Z}^d$-SFTs. Interestingly, they show that for $d \geq 2$ the characterization is of an algorithmic nature. More precisely, the numbers attained as entropies of $\mathbb{Z}^d$-SFTs coincides with the set of non-negative upper semi-computable real numbers. Their classification relies on a construction which embeds arbitrarily large computation diagrams of an arbitrary Turing machine into a $\mathbb{Z}^d$-SFT.

The purpose of this study is to explore what entropies can be achieved by subshifts of finite type defined on an arbitrary amenable group G. In particular, we shall present a way to transfer entropies attainable by SFTs on a group H to G whenever H can be "geometrically embedded into G". A simple observation is that whenever H is a subgroup of an amenable group G, then any number obtained as the topological entropy of an H-SFT X can also be obtained as the topological entropy of a G-SFT Y. Indeed, this is achieved by letting Y be the set of all configurations such that every H-coset contains a configuration of X and there are no restrictions between each individual H-coset.

In this article we generalize the above construction introducing the notion of group charts. A group chart (X, γ) is a dynamical structure consisting of a dynamical system $G \curvearrowright X$ and a continuous cocycle $\gamma\colon H \times X \to G$ that associates configurations in X to partitions of its underlying group G into quotients of H. Whenever X is a G-subshift, we can use the partitions induced by the chart (X, γ) to embed any H-subshift Y into a G-subshift $Y_\gamma[X]$ which stores the information of Y in a natural way. We shall show (Theorem 3.10) that for any such embedding in which the cocycle induces free actions, the topological entropy satisfies the following addition formula,

$$h_{\text{top}}(G \curvearrowright Y_\gamma[X]) = h_{\text{top}}(G \curvearrowright X) + h_{\text{top}}(H \curvearrowright Y).$$

Furthermore, if both X and Y are SFTs, we have that $Y_\gamma[X]$ is an SFT. Therefore this formula can be used to embed the entropies of H-SFTs into the set of entropies of G-SFTs up to a fixed additive constant. We shall introduce the notion of group charts and give a proof of the addition formula on Section 3.1.

In Section 3.2 we shall show that whenever a group chart is given by a G-SFT X, then we can choose it in such a way that its entropy is arbitrarily small (Corollary 3.20). This will follow from a theorem that gives a canonical way of reducing the entropy of subshifts of finite type defined on an arbitrary countable amenable groups (Theorem 3.17). We shall prove this theorem using the theory of quasitilings introduced by Ornstein and Weiss [21] and a recent result of Downarowicz, Huczec and Zhang [10].

In Section 3.3 we will characterize the existence of free charts, that is, charts for which every element of x codes a true partition of G into copies of H, through the notion of translation-like actions introduced by Whyte [26]. Furthermore, following the ideas of Jeandel [15], we shall show that whenever H is finitely presented and there exists a non-empty H-SFT on which H acts freely, then one can always find a free chart (X, γ) for which X is a G-SFT. Putting all of the previous results together, we shall show the following result.

Theorem 3.24. *Let G, H be finitely generated amenable groups and let $\mathcal{E}_{SFT}(H)$ and $\mathcal{E}_{SFT}(G)$ respectively denote the set of real numbers attainable as topological entropies of an SFT in each group. Suppose that*

1. *H admits a translation-like action on G.*
2. *H is finitely presented.*
3. *There exists a non-empty H-SFT for which the H-action is free.*

Then, for every $\varepsilon > 0$ there exists a G-SFT X such that $h_{top}(G \curvearrowright X) < \varepsilon$ and

$$h_{top}(G \curvearrowright X) + \mathcal{E}_{SFT}(H) \subset \mathcal{E}_{SFT}(G).$$

In Section 4 we shall apply the above theorem to study the groups on which $\mathbb{Z}^2$ acts translation-like. It shall follow that modulo a computability obstruction, any finitely generated amenable group on which $\mathbb{Z}^2$ acts translation-like admits the same characterization of the set of numbers that can be attained as topological entropies of subshifts of finite type as $\mathbb{Z}^2$. Namely,

Theorem 4.7. *Let G be a finitely generated amenable group with decidable word problem which admits a translation-like action by $\mathbb{Z}^2$. The set of entropies attainable by G-subshifts of finite type is the set of non-negative upper semi-computable numbers.*

Finally, in Section 5 we shall use Theorem 4.7 to give a characterization of the numbers attainable as topological entropies of subshifts of finite type in several classes of groups. More precisely, we shall give a complete classification for polycyclic-by-finite groups (Theorem 5.6), products of two infinite and finitely generated amenable groups with decidable word problem (Corollary 5.10), countable amenable groups which admit a presentation with decidable word problem and a finitely generated subgroup on which $\mathbb{Z}^2$ acts translation-like (Corollary 5.12) and infinite and finitely generated amenable branch groups with decidable word problem (Theorem 5.16).

2 Preliminaries and notation

In this note we shall consider left actions $G \curvearrowright X$ of countable amenable groups G over compact metric spaces X by homeomorphisms. Let us denote by $F \Subset G$ a finite subset of G and by 1_G the identity of G. For $K \Subset G$ and $\varepsilon > 0$ we say that $F \Subset G$ is left (K, ε)-invariant if $|KF \triangle F| \leq \varepsilon |F|$. From this point forward we shall omit the word left and plainly speak about (K, ε)-invariant sets. A sequence $\{F_n\}_{n \in \mathbb{N}}$ of finite subsets of G is called a Følner sequence if for every $K \Subset G$ and $\varepsilon > 0$ the sequence is eventually (K, ε)-invariant.

2.1 Shift spaces

Let Σ be a finite set and G be a group. The set $\Sigma^G = \{x \colon G \to \Sigma\}$ equipped with the left group action $G \curvearrowright X$ given by $gx(h) \triangleq x(hg)$ is the **full G-shift**. The elements $a \in \Sigma$ and $x \in \Sigma^G$ are called **symbols** and **configurations** respectively. We endow Σ^G with the product topology generated by the clopen subbase given by the **cylinders** $[a]_g \triangleq \{x \in \Sigma^G \mid x(g) = a\}$. A **support** is a finite subset $F \Subset G$. Given a support F, a **pattern** with support F is an element $p \in \Sigma^F$ and we write $\mathrm{supp}(p) = F$. We denote the cylinder generated by p by $[p] = \bigcap_{h \in F} [p(h)]_h$.

A subset $X \subset \Sigma^G$ is a **G-subshift** if and only if it is G-invariant and closed in the product topology. Equivalently, X is a G-subshift if and only if there exists a set of forbidden patterns $\mathcal{F}$ such that

$$X = X_{\mathcal{F}} \triangleq \Sigma^G \setminus \bigcup_{p \in \mathcal{F}, g \in G} g[p].$$

Given a subshift $X \subset \Sigma^G$ and a support $F \Subset G$ the **language with support** F is the set $L_F(X) = \{p \in \Sigma^F \mid [p] \cap X \neq \varnothing\}$ of all patterns which appear in some configuration $x \in X$. The **language** of X is the set $L(X) = \bigcup_{F \Subset G} L_F(X)$.

Remark 2.1. It is also possible to define the left G-action by $gx(h) \triangleq x(g^{-1}h)$ instead of $x(hg)$. In this article we chose the latter in order to minimize the amount of superindices $^{-1}$ and to make the notation compatible with the setting of [10], whose results we shall use to prove Theorem 3.17.

Definition 2.2. *We say that a subshift X is of **finite type (SFT)** if there exists a finite set $\mathcal{F}$ of forbidden patterns such that $X = X_{\mathcal{F}}$.*

2.2 Topological entropy

Let $G \curvearrowright X$ be the action of a group over a compact metrizable space by homeomorphisms. Given two open covers $\mathcal{U}, \mathcal{V}$ of X we define their **join** by $\mathcal{U} \vee \mathcal{V} = \{U \cap V \mid U \in \mathcal{U}, V \in \mathcal{V}\}$. For $g \in G$ let $g\mathcal{U} = \{gU \mid U \in \mathcal{U}\}$ and denote by $N(\mathcal{U})$ the smallest cardinality of a subcover of $\mathcal{U}$. If F is a finite subset of G, denote by $\mathcal{U}^F$ the join

$$\mathcal{U}^F = \bigvee_{g \in F} g^{-1}\mathcal{U}.$$

Definition 2.3. *Let $G \curvearrowright X$ be the action of a countable amenable group, $\mathcal{U}$ an open cover and $\{F_n\}_{n \in \mathbb{N}}$ a Følner sequence for G. We define the **topological entropy of $G \curvearrowright X$ with respect to** $\mathcal{U}$ as*

$$h_{top}(G \curvearrowright X, \mathcal{U}) = \lim_{n \to \infty} \frac{1}{|F_n|} \log N(\mathcal{U}^{F_n}).$$

The function $F \mapsto \log N(\mathcal{U}^F)$ is subadditive and thus the limit does not depend on the choice of Følner sequence, see for instance [21, 17]. The **topological entropy** of $G \curvearrowright X$ is defined as

$$h_{\mathrm{top}}(G \curvearrowright X) = \sup_{\mathcal{U}} h_{\mathrm{top}}(G \curvearrowright X, \mathcal{U}).$$

In the case where $G \curvearrowright X$ is expansive, any open cover $\mathcal{U}$ whose elements have diameter less than the expansivity constant achieves the supremum. Particularly, in the case of a subshift $X \subset \Sigma^G$ we may consider the partition $\xi = \{[a]_{1_G} \mid a \in \Sigma\}$. For a finite $F \subset G$ we obtain that $\xi^F = \{[p] \mid p \in L_F(X)\}$. Hence, whenever X is a subshift its topological entropy can be computed by

$$h_{\mathrm{top}}(G \curvearrowright X) = \lim_{n \to \infty} \frac{1}{|F_n|} \log(|L_{F_n}(X)|).$$

A more intuitive way to understand this limit, is that the function $F \mapsto \frac{1}{|F|} \log(|L_F(X)|)$ converges as F becomes more and more invariant, that is, for every $\varepsilon > 0$ there exists $K \Subset G$ and $\delta > 0$ such that for any (K,δ)-invariant set F we have $|h_{\text{top}}(G \curvearrowright X) - \frac{1}{|F|}\log(|L_F(X)|)| \leq \varepsilon$. For a self contained proof and relevant background see [16, Theorem 4.38].

In the case when the open cover $\mathcal{U}$ consists of pairwise disjoint open sets, it can be shown that the function $F \mapsto \log N(\mathcal{U}^F)$ is not only subadditive, but satisfies Shearer's inequality (see [9, Corollary 6.2]). This in turn implies that in the case of a subshift we may write:

$$h_{\text{top}}(G \curvearrowright X) = \inf_{F \in \mathcal{F}(G)} \frac{1}{|F|}\log(|L_F(X)|). \tag{1}$$

where $\mathcal{F}(G)$ denotes the set of all finite subsets of G, see [9, Corollary 6.3]).

Remark 2.4. In fact the result that topological entropy can be computed as an infimum over all finite subsets holds for any $G \curvearrowright X$, although it may not hold individually for every partition $\mathcal{U}$. This was proven in [9] using the variational principle. A good way to think about it is that in the context of amenable groups, the topological entropy coincides with the naive entropy of Burton [6].

Let us introduce the following notation which will be useful in the remainder of the article. For a group G, we denote the set of real numbers attained as topological entropies of G-SFTs by $\mathcal{E}_{\text{SFT}}(G)$.

$$\mathcal{E}_{\text{SFT}}(G) = \{r \in \mathbb{R} \mid \text{there exists a } G\text{-SFT } X, h_{\text{top}}(G \curvearrowright X) = r\}$$

Let us state two classical theorems from the literature which will be used further on. Recall that a **Perron number** is a real algebraic integer greater than 1 and greater than the modulus of its algebraic conjugates.

Theorem 2.5 (Lind [20])**.** *$\mathcal{E}_{SFT}(\mathbb{Z})$ is the set of non-negative rational multiples of logarithms of Perron numbers.*

In order to state the second result, we need to introduce the notion of upper semi-computable numbers, they are also sometimes called "right-recursively enumerable numbers".

Definition 2.6. *A real number r is **upper semi-computable** if there exists a Turing machine T which on input $n \in \mathbb{N}$ halts with the coding of a rational number $q_n \geq r$ on its tape such that $\lim_{n\to\infty} q_n = r$.*

Theorem 2.7 (Hochman and Meyerovitch [14])**.** *For $d \geq 2$, $\mathcal{E}_{SFT}(\mathbb{Z}^d)$ is the set of non-negative upper semi-computable numbers.*

3 Realization of entropies of subshifts of finite type

3.1 Group charts and the addition formula

Definition 3.1. *Let G, H be two topological groups and let X be a compact topological space on which G acts on the left by homeomorphisms. A continuous map $\gamma\colon H \times X \to G$ is called an H**-cocycle** if it satisfies the equation*

$$\gamma(h_1h_2, x) = \gamma(h_1, \gamma(h_2,x)x) \cdot \gamma(h_2, x) \text{ for every } h_1, h_2 \text{ in } H.$$

The cocycle equation can be represented by the diagram shown on Figure 1. Let us clarify how this equation fits within the classical setting of cocycles. A continuous map γ as above induces an action $H \curvearrowright X$ by setting $h \cdot x = \gamma(h,x)x$, where the product on the right is the one associated to the action $G \curvearrowright X$. With this action $H \curvearrowright X$ in mind, the equation simplifies to the better known equation for cocycles

$$\gamma(h_1h_2, x) = \gamma(h_1, h_2 \cdot x) \cdot \gamma(h_2, x) \text{ for every } h_1, h_2 \text{ in } H.$$

Any H-cocycle γ induces a family $\{H \overset{x}{\curvearrowright} G\}_{x \in X}$ of left H-actions on G. Indeed, if for fixed $x \in X$ we define for $h \in H$ and $g \in G$, the action given by $h \cdot_x g \triangleq \gamma(h, gx)g$, then for all $h_1, h_2 \in H$ we have

$$\begin{aligned}
(h_1h_2) \cdot_x g &= \gamma(h_1h_2, gx)g \\
&= (\gamma(h_1, \gamma(h_2,gx)gx) \cdot \gamma(h_2,gx))g \\
&= \gamma(h_1, (\gamma(h_2,gx)g)x) \cdot (\gamma(h_2,gx)g) \\
&= h_1 \cdot_x (\gamma(h_2,gx)g) \\
&= h_1 \cdot_x (h_2 \cdot_x g).
\end{aligned}$$

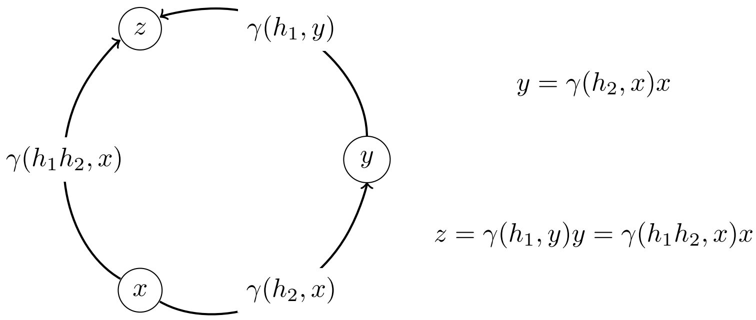

Figure 1: The circles x, y, z represent points in the space X while the arrows represent left multiplication by group elements. The cocycle equation states that the arrows commute: $\gamma(h_1h_2, x) = \gamma(h_1, y)\gamma(h_2, x)$.

Remark 3.2. If H is a finitely generated group and S a finite generating set for H, then the values of any H-cocycle γ restricted to $S \times X$ define γ completely. Furthermore, whenever G is countable, by continuity of γ and compactness of $S \times X$ we have that γ must be uniformly bounded on $S \times X$ and thus $\gamma(S \times X) \Subset G$. Hence if X is a G-subshift, there exists a finite set $F \Subset G$ such that γ restricted to $S \times X$ is completely defined by a finite map $\tilde{\gamma}\colon S \times L_F(X)$.

The following notion is strongly motivated by the work of Jeandel [15].

Definition 3.3. *Let G, H be two countable groups. Given a left action $G \curvearrowright X$ and an H-cocycle $\gamma\colon H \times X \to G$ we say the pair (X, γ) is a **G-chart** of H. Furthermore, if for each $x \in X$ the action $H \overset{x}{\curvearrowright} G$ is free, we say that (X, γ) is a **free G-chart** of H.*

Example 3.4. The trivial system $G \curvearrowright \{0\}$ consisting of a single point and the cocycle $\gamma\colon H \times \{0\} \to G$ which sends $(h, 0) \mapsto h$ is a free G-chart of H for any subgroup $H \leq G$. $\diamond$

Example 3.5. Let $G = \mathbb{Z}^2$ and let $\Sigma_{\texttt{snake}}$ be the set of vector pairs given by

$$\Sigma_{\texttt{snake}} = \{(\ell, r) \in \{(1, 0), (-1, 0), (0, 1), (0, -1)\}^2 \mid \ell \neq r\}$$

Visually, we may represent $\Sigma_{\texttt{snake}}$ by the set of square unit tiles shown on Figure 2. The first vector is represented by the tail of the arrow and the second vector by the outgoing arrow.

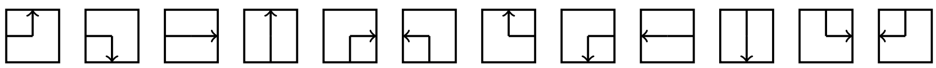

Figure 2: The alphabet $\Sigma_{\texttt{snake}}$.

For $a = (\ell, r) \in \Sigma_{\texttt{snake}}$ let $L(a) = \ell$ and $R(a) = r$. We define the **snake shift** as the $\mathbb{Z}^2$-SFT $X_{\texttt{snake}} \subset (\Sigma_{\texttt{snake}})^{\mathbb{Z}^2}$ of all configurations x such that for every position $v \in \mathbb{Z}^2$, we have $R(x(v)) = L(x(v + R(x(v))))$ and $L(x(v)) = R(x(v + L(x(v))))$. Visually, these are the configurations such that every outgoing arrow matches with an incoming arrow. Let $\gamma_{\texttt{snake}}\colon \mathbb{Z} \times X \to \mathbb{Z}^2$ be the $\mathbb{Z}$-cocycle defined by $\gamma_{\texttt{snake}}(1, x) = R(x((0, 0)))$ and $\gamma_{\texttt{snake}}(-1, x) = L(x((0, 0)))$. It can be verified that $(X_{\texttt{snake}}, \gamma_{\texttt{snake}})$ is a $\mathbb{Z}^2$-chart of $\mathbb{Z}$.

The $\mathbb{Z}^2$-chart $(X_{\texttt{snake}}, \gamma_{\texttt{snake}})$ of $\mathbb{Z}$ is not free. Indeed, every configuration x in which a cycle appears induces an action $\mathbb{Z} \overset{x}{\curvearrowright} \mathbb{Z}^2$ which is not free. Let $X^{\texttt{free}}_{\texttt{snake}} \subset X_{\texttt{snake}}$ be the **free snake** subshift consisting of all configurations $x \in X_{\texttt{snake}}$ such that no cycles appear, it can be verified that $(X^{\texttt{free}}_{\texttt{snake}}, \gamma_{\texttt{snake}}|_{\mathbb{Z} \times X^{\texttt{free}}_{\texttt{snake}}})$ is a free $\mathbb{Z}^2$-chart of $\mathbb{Z}$. See Figure 3. $\diamond$

Let Σ be a set. The notion of G-chart gives canonical way to recover an H-orbit of Σ given a G-orbit $y \in \Sigma^G$ and basepoints $x \in X$ and $g \in G$. Indeed, if (X, γ) is a G-chart of H we can associate to every $y \in \Sigma^G$ an orbit $\pi_{x,g}(y) \in \Sigma^H$ by setting

$$\pi_{x,g}(y)(h) \triangleq y(h \cdot_x g) = y(\gamma(h, gx)g) \text{ for every } h \text{ in } H.$$

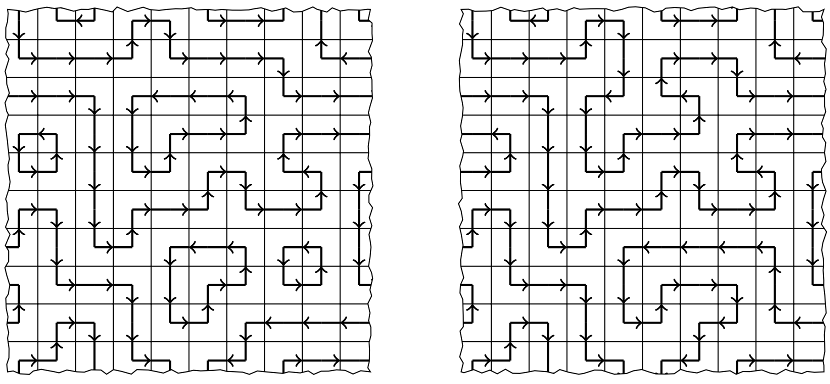

Figure 3: On the left we see a local patch of $X_{\texttt{snake}}$. The value of the $\mathbb{Z}$-cocycle $\gamma_{\texttt{snake}}(n, x)$ corresponds to the vector of $\mathbb{Z}^2$ obtained by following the arrow at the origin n times. On the right we see a local patch of a configuration of $X^{\texttt{free}}_{\texttt{snake}}$. As cycles are forbidden, the cocycle induces a free action.

Moreover, this configuration satisfies that for every $h_1, h_2 \in H$:

$$\begin{aligned}(h_2\pi_{x,g}(y))(h_1) &= \pi_{x,g}(y)(h_1h_2)\\ &= y((h_1h_2)\cdot_x g)\\ &= y(h_1 \cdot_x (h_2 \cdot_x g))\\ &= (\pi_{x,h_2\cdot_x g}(y))(h_1)\end{aligned}$$

In other words, the left shift action of h_2 on $\pi_{x,g}(y)$ is the same as $\pi_{x,h_2\cdot_x g}(y)$, that is, the configuration obtained by changing the basepoint g by $h_2 \cdot_x g$.

From now on, we shall only consider G-charts (X, γ) where X is a G-subshift.

Definition 3.6. *Let (X, γ) be a G-chart of H and $Y \subset \Sigma^H$ be an H-subshift. The (X, γ)-**embedding of** Y is the G-subshift $Y_\gamma[X] \subset \Sigma^G \times X$ which has the property that $(y, x) \in Y_\gamma[X]$ if and only if for every $g \in G$ then $\pi_{x,g}(y)$ is in Y.*

In simpler words, $Y_\gamma[X]$ is the subshift of all pairs (y, x) where $x \in X$ and every copy of H induced by the action $H \overset{x}{\curvearrowright} G$ is decorated independently with a configuration from Y.

Example 3.7. Consider the free $\mathbb{Z}^2$-chart (X, γ) of $\mathbb{Z}$ from Example 3.5, that is, $X = X^{\texttt{free}}_{\texttt{snake}}$ and $\gamma = \gamma_{\texttt{snake}}|_{\mathbb{Z}\times X^{\texttt{free}}_{\texttt{snake}}}$. Consider the $\mathbb{Z}$-subshift Y consisting on the orbit of the sequence x over the alphabet $\Sigma = \{\square, \triangle, \bigcirc\}$ given by

$$x(n) = \begin{cases} \square & \text{if } n = 0 \bmod 3\\ \triangle & \text{if } n = 1 \bmod 3\\ \bigcirc & \text{if } n = 2 \bmod 3.\end{cases}$$

The subshift $Y_\gamma[X]$ is the set of all configurations $(y, x) \in \{\square, \triangle, \bigcirc\}^{\mathbb{Z}^2} \times X$ such that every path in $x \in X$ induced by γ is decorated independently with a configuration from Y, see Figure 4. $\diamond$

Remark 3.8. Let (X, γ) be a G-chart of H. If X is a G-SFT and Y is an H-SFT then $Y_\gamma[X]$ is also a G-SFT.

Remark 3.9. If (X, γ) is a G-chart of H and there is $x \in X$ such that $H \overset{x}{\curvearrowright} G$ is free, then the map $\pi\colon Y_\gamma[X] \to Y$ given by $\pi(y, x) = \pi_{x,1_G}(y)$ is surjective. In particular, $Y_\gamma[X]$ is non-empty if and only if Y is non-empty.

The following result is the main tool that will allow us to take a subshift of finite type with fixed topological entropy defined on a group H, and realize it, modulo a fixed constant, as the topological entropy of a subshift of finite type defined on any group where H can be freely charted . It shows that the entropy of any subshift which is embedded in a free chart can be expressed through an addition formula.

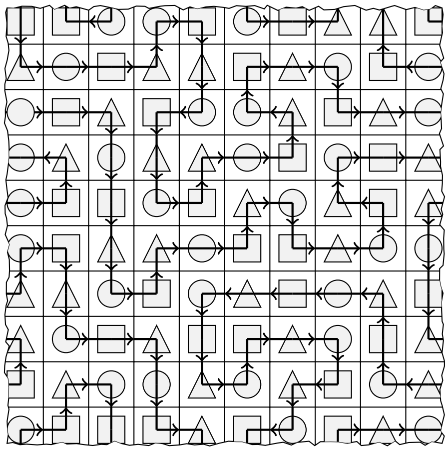

Figure 4: The subshift $Y_\gamma[X]$ is obtained by "overlaying" the copies of H induced by γ on X with configurations of Y.

Theorem 3.10 (addition formula)**.** *Let G, H be countable amenable groups. For any free G-chart (X, γ) of H and for any H-subshift Y we have*

$$h_{top}(G \curvearrowright Y_\gamma[X]) = h_{top}(H \curvearrowright Y) + h_{top}(G \curvearrowright X). \tag{2}$$

Proof. Denote by Σ_X and Σ_Y the alphabets of X and Y respectively. Let $\varepsilon > 0$. There exists $S \Subset H$ and $\delta > 0$ such that every non-empty (S, δ)-invariant set $F \Subset H$ satisfies

$$\mathrm{e}^{h_{\mathrm{top}}(H \curvearrowright Y)|F|} \leq |L_F(Y)| \leq \mathrm{e}^{(h_{\mathrm{top}}(H \curvearrowright Y)+\varepsilon)|F|}.$$

Let $\gamma \colon H \times X \to G$ be the H-cocycle associated to X. As G is countable and S is finite, the restriction of γ to $S \times X$ is bounded. Let $W_1 \Subset G$ be a set such that $\gamma(S \times X) \subset W_1$. By continuity of γ there exists $W_2 \Subset G$ such that a set such that for every $s \in S$ we have $\gamma(s, x) = \gamma(s, y)$ whenever $x|_{W_2} = y|_{W_2}$. For every $\varepsilon' > 0$ there exists a finite set $S' \supset W_1 \cup W_2$ and $\delta' > 0$ such that every non-empty (S', δ')-invariant $F' \Subset G$ satisfies

$$\mathrm{e}^{h_{\mathrm{top}}(G \curvearrowright X)|F'|} \leq |L_{F'}(X)| \leq \mathrm{e}^{(h_{\mathrm{top}}(G \curvearrowright X)+\varepsilon')|F'|}.$$

Let $F' \Subset G$ be an (S', δ')-invariant set and consider a pattern $p \in L_{S'F'}(X)$. As $W_2 \subset S'$, for each $f' \in F'$ and $s \in S$ the map $\gamma_p(s, f') \triangleq \gamma(s, f'x)$ where $x \in X$ is any configuration such that $x|_{S'F'} = p$ is well defined. Let us define the relation $R \subset F' \times F'$ as the smallest equivalence relation such that whenever $f_1', f_2' \in F'$ satisfy that for some $s_1, s_2 \in S$ we have $\gamma_p(s_1, f_1')f_1' = \gamma_p(s_2, f_2')f_2'$, then $(f_1', f_2') \in R$.

The equivalence relation R induces a partition $F' = F_1^p \uplus F_2^p \uplus \cdots \uplus F_{k(p)}^p$. Let us denote by $\partial_S F_i^p$ the set of all $g' \in S'F' \setminus F'$ for which there is $f' \in F_i^p$ and $s \in S$ such that $\gamma_p(s, f')f' = g'$. By definition of R, note that the sets $\partial_S F_i^p$ are pairwise disjoint and $\partial_S F_i^p \subset S'F' \setminus F'$.

We obtain that $\sum_{i=1}^{k(p)} |\partial_S F_i^p| \leq |S'F' \setminus F'| \leq \delta'|F'|$. Dividing both sides by $|F'|$ and multiplying each left term by $\frac{|F_i^p|}{|F_i^p|}$ we obtain:

$$\sum_{i=1}^{k(p)} \frac{|\partial_S F_i^p|}{|F_i^p|} \frac{|F_i^p|}{|F'|} \leq \delta'.$$

Denote by μ_i the ratio $\mu_i = \frac{|F_i^p|}{|F'|}$ and by $\delta_i = \frac{|\partial_S F_i^p|}{|F_i^p|}$. Note that $\mu_i \in [0, 1]$, $\sum_{i=1}^{k(p)} \mu_i = 1$ and $\delta_i \in [0, |S|]$. Let $I(p)$ be the set of indices such that $\delta_i \leq \delta$. We have that $\sum_{i \in I(p)} \delta_i \mu_i + \sum_{j \notin I(p)} \delta_j \mu_j \leq \delta'$. A simple manipulation of this expression yields

$$\sum_{i \in I(p)} \mu_i \geq 1 - \frac{\delta'}{\delta}. \tag{3}$$

The intuitive meaning of Equation (3) is that the total amount of sites in the (S', ε')-invariant set F' which lie in an induced subset of H which is (S,δ)-invariant can be made arbitrarily large by tweaking the ratio $\frac{\delta'}{\delta}$.

As the G-chart (X,γ) of H is free, we can identify each set F_i^p with a subset $H_i^p \Subset H$ and $\partial_S F_i^p$ with $SH_i^p \setminus H_i^p$. Furthermore, we have $|H_i^p| = |F_i^p|$ and $|SH_i^p \setminus H_i^p| = |\partial_S F_i^p|$. In other words, whenever $|\partial_S F_i^p| \leq \delta |F_i^p|$, the set H_i is (S,δ)-invariant. Now we use this computation to estimate the size of $|L_{F'}(Y_\gamma[X])|$. Clearly $|L_{S'F'}(X)| \geq |L_{F'}(X)|$, we can thus obtain

$$\begin{aligned}
|L_{F'}(Y_\gamma[X])| &\leq \sum_{p \in L_{S'F'}(X)} \prod_{i=1}^{k(p)} |L_{H_i^p}(Y)| \\
&\leq \sum_{p \in L_{S'F'}(X)} \prod_{i \in I(p)} |L_{H_i^p}(Y)| \prod_{j \notin I(p)} |L_{H_j^p}(Y)| \\
&\leq \sum_{p \in L_{S'F'}(X)} \prod_{i \in I(p)} |L_{H_i^p}(Y)| \prod_{j \notin I(p)} |\Sigma_Y|^{|H_j^p|} \\
&\leq |\Sigma_Y|^{\frac{\delta'|F'|}{\delta}} \sum_{p \in L_{S'F'}(X)} \prod_{i \in I(p)} |L_{H_i^p}(Y)|.
\end{aligned}$$

As each H_i^p for $i \in I(p)$ is (S,δ)-invariant, we get $|L_{H_i^p}(Y)| \leq \mathrm{e}^{(h_{\mathrm{top}}(H \curvearrowright Y)+\varepsilon)|H_i|}$ and thus

$$\begin{aligned}
|L_{F'}(Y_\gamma[X])| &\leq |\Sigma_Y|^{\frac{\delta'|F'|}{\delta}} \sum_{p \in L_{S'F'}(X)} \prod_{i \in I(p)} \mathrm{e}^{(h_{\mathrm{top}}(H \curvearrowright Y)+\varepsilon)|H_i|} \\
&\leq |\Sigma_Y|^{\frac{\delta'|F'|}{\delta}} \sum_{p \in L_{S'F'}(X)} \mathrm{e}^{(h_{\mathrm{top}}(H \curvearrowright Y)+\varepsilon)|F'|(1-\frac{\delta'}{\delta})} \\
&\leq |\Sigma_Y|^{\frac{\delta'|F'|}{\delta}} \mathrm{e}^{(h_{\mathrm{top}}(H \curvearrowright Y)+\varepsilon)|F'|} |L_{S'F'}(X)|.
\end{aligned}$$

Therefore we obtain that,

$$\begin{aligned}
\frac{1}{|F'|}\log(|L_{F'}(Y_\gamma[X])|) &\leq \frac{\delta'}{\delta}\log(|\Sigma_Y|) + (h_{\mathrm{top}}(H \curvearrowright Y)+\varepsilon) + \frac{1}{|F'|}\log(|L_{S'F'}(X)|) \\
&\leq \frac{\delta'}{\delta}\log(|\Sigma_Y|) + h_{\mathrm{top}}(H \curvearrowright Y)+\varepsilon + \frac{1}{|F'|}\big(\log(|L_{F'}(X)|) + \log(|L_{S'F' \setminus F'}(X)|)\big) \\
&\leq \frac{\delta'}{\delta}\log(|\Sigma_Y|) + h_{\mathrm{top}}(H \curvearrowright Y)+\varepsilon + \frac{1}{|F'|}\log(|L_{F'}(X)|) + \frac{|S'F' \setminus F'|}{|F'|}\log(|\Sigma_X|).
\end{aligned}$$

As F' is an (S',δ')-invariant set, we get that $|S'F' \setminus F'| \leq \delta'|F'|$. Furthermore, by definition this also implies that $\log(|L_{F'}(X)| \leq (h_{\mathrm{top}}(G \curvearrowright X)+\varepsilon')|F'|$, therefore for every (S',δ')-invariant set F' we have,

$$\frac{1}{|F'|}\log(|L_{F'}(Y_\gamma[X])|) \leq h_{\mathrm{top}}(H \curvearrowright Y) + h_{\mathrm{top}}(G \curvearrowright X) + \varepsilon + \varepsilon' + \frac{\delta'}{\delta}\log(|\Sigma_Y|) + \delta'\log(|\Sigma_X|).$$

By the infimum formula for the entropy, the previous expression is an upper bound for the entropy $h_{\mathrm{top}}(G \curvearrowright Y_\gamma[X])$. Now choose $\varepsilon = \varepsilon' = \frac{1}{n}$, this bounds the available values of δ and δ' above. We may arbitrarily choose $\delta' \leq \frac{\delta}{n}$. Letting n go to infinity we obtain,

$$h_{\mathrm{top}}(G \curvearrowright Y_\gamma[X]) \leq h_{\mathrm{top}}(H \curvearrowright Y) + h_{\mathrm{top}}(G \curvearrowright X). \tag{4}$$

For the lower bound, Equation (1) shows that a lower bound for $L_{H_i^p}(Y)$ is given by $\mathrm{e}^{h_{\mathrm{top}}(H \curvearrowright Y)|H_i^p|}$. It is then not hard to see that for every $F' \Subset G$ we have,

$$|L_{F'}(Y_\gamma[X])| \geq |L_{F'}(X)|\mathrm{e}^{h_{\mathrm{top}}(H \curvearrowright Y)|F'|}. \tag{5}$$

From where we obtain the other inequality. □

Recall that by Remark 3.8 if both the subshift X in a chart (X,γ) and the embedded subshift Y are SFTs, then $Y_\gamma[X]$ is also an SFT. This gives us a way of producing a new G-SFT those topological entropy is the sum of their entropies.

Corollary 3.11. *If (X,γ) is a free G-chart of H and X is a G-SFT, then for every H-SFT Y there exists a G-SFT Z which has entropy $h_{top}(G \curvearrowright X) + h_{top}(H \curvearrowright Y)$. In other words,*

$$h_{top}(G \curvearrowright X) + \mathcal{E}_{SFT}(H) \subset \mathcal{E}_{SFT}(G).$$

In what follows we shall show that if there is at least one free G-chart (X,γ) of H where X is a G-SFT, then it is always possible to find another such chart where X can have arbitrarily low entropy.

3.2 Reducing the entropy of a chart

The goal of this section is to develop a method for reducing the entropy of a subshift of finite type in such a way that the new subshift of finite type preserves any cocycle defined on the original one. In order to do this we will use the machinery of quasitilings developed by Ornstein and Weiss in [21]. In order to minimize the complexity of the proof, we shall in fact use a recent result by Downarowicz, Huczec and Zhang [10] which shows, for any countable amenable group, the existence of zero-entropy exact tilings where each tile can be made arbitrarily invariant.

The ideas presented in this section have been strongly influenced by the work of Frisch and Tamuz [12] which use similar methods to study generic properties of the set of all subshifts.

Definition 3.12. *Let G be a group. A* ***tile set*** *is a finite collection $\mathcal{T} = \{T_1,\dots,T_n\}$ of finite subsets of G which contain the identity. A* ***tiling*** *of G by $\mathcal{T}$ is a function $\tau\colon G \to \mathcal{T} \cup \{\varnothing\}$ such that:*

1. *(τ is pairwise-disjoint) For every $g,h \in G$, if $g \neq h$ then $\tau(g)g \cap \tau(h)h = \varnothing$.*
2. *(τ covers G) For every $g \in G$ there exists $h \in G$ such that $g \in \tau(h)h$.*

Lemma 3.13. *Let $\mathcal{T}$ be a tileset. The collection of all tilings of G by $\mathcal{T}$ is a G-SFT.*

Proof. Let $X_{\mathcal{T}} \subset (\mathcal{T} \cup \{\varnothing\})^G$ be the set of all configurations $\tau\colon G \to \mathcal{T} \cup \{\varnothing\}$ which avoid the set of forbidden patterns $\mathcal{D} \cup \mathcal{C}$ where

1. $\mathcal{D}$ is the set of all patterns p with support $\{1_G, g\}$ where $g = t_2^{-1}t_1 \neq 1_G$ for some $t_1, t_2 \in \bigcup_{i\leq n} T_i$ and which satisfy that $p(1_G) \cap p(g)g \neq \varnothing$.
2. $\mathcal{C}$ consists of all patterns q with support $\bigcup_{i\leq n} T_i^{-1}$ such that $1_G \notin q(g)g$ for every $g \in \operatorname{supp}(q)$.

Both $\mathcal{D}$ and $\mathcal{C}$ are finite and thus $X_{\mathcal{T}}$ is a G-SFT. We claim that $\tau \in X_{\mathcal{T}}$ if and only if τ is a tiling of G by $\mathcal{T}$. We shall show this in two parts. Let $\tau \in (\mathcal{T} \cup \{\varnothing\})^G$.

1. τ is pairwise disjoint if and only if no pattern from $\mathcal{D}$ appears in τ. Indeed, if τ is not pairwise disjoint there are $h_1 \neq h_2$ such that $\tau(h_1)h_1 \cap \tau(h_2)h_2 \neq \varnothing$. Letting $\tau' = h_1\tau$ we have $\tau'(1_G) = \tau(h_1)$ and $\tau'(h_2h_1^{-1}) = \tau(h_2)$, therefore $\tau(h_1)h_1 \cap \tau(h_2)h_2 \neq \varnothing$ if and only if $\tau'(1_G) \cap (\tau'(h_2h_1^{-1}))h_2h_1^{-1} \neq \varnothing$. This means that there exist $t_1 \in \tau'(1_G)$ and $t_2 \in \tau'(h_2h_1^{-1})$ such that $t_1 = t_2h_2h_1^{-1}$, equivalently such that $h_2h_1^{-1} = t_2^{-1}t_1$. Let $g = t_2^{-1}t_1$. We get that $\tau'(1_G) \cap (\tau'(g))g \neq \varnothing$ if and only if $\tau'|_{\{1_G,g\}} \in \mathcal{D}$ and thus $\tau'|_{\{1_G,g\}}$ appears in τ.
2. τ covers G if and only if no pattern from $\mathcal{C}$ appears in τ. Indeed, suppose τ does not cover G, then there is $g \in G$ such that for every $h \in G$, $g \notin \tau(h)h$. Letting $\tau' = g\tau$ we obtain that $\tau(h) = \tau'(hg^{-1})$ and hence $g \notin \tau(h)h$ for every $h \in G$ if and only if $1_G \notin \tau'(hg^{-1})hg^{-1}$ for every $h \in G$ which is the same as saying that $1_G \notin \tau'(s)s$ for every $s \in G$. This is equivalent to $\tau'|_{\bigcup_{i\leq n} T_i^{-1}} \in \mathcal{C}$. Therefore τ does not cover G if and only if there is $g \in G$ such that $(g\tau)|_{\bigcup_{i\leq n} T_i^{-1}} \in \mathcal{C}$, which is the same as saying that a pattern from $\mathcal{C}$ appears in τ.

Therefore $\tau \in X_{\mathcal{T}}$ if and only if τ is a tiling of G by $\mathcal{T}$. $\square$

Remark 3.14. The orbit closure of any tiling $\tau\colon G \to \mathcal{T} \cup \{\varnothing\}$ forms a G-subshift which is not necessarily of finite type. We shall denote by $h_{\mathrm{top}}(\tau)$ the topological entropy of said subshift.

Theorem 3.15 (Downarowicz, Huczek and Zhang [10])**.** *Let G be a countable amenable group. For any $F \Subset G$ and $\delta > 0$ there exists a tile set $\mathcal{T}$ such that every $T \in \mathcal{T}$ is (F,δ)-invariant and there exists a tiling τ by $\mathcal{T}$ such that $h_{top}(\tau) = 0$.*

Lemma 3.16. *Let G be a countable amenable group and $X \subset \Sigma^G$ be a G-SFT. Suppose that $Y \subset X$ is a subshift of X, then for every $\varepsilon > 0$ there exists a G-SFT $Z \subset X$ so that*

$$h_{top}(G \curvearrowright Y) \leq h_{top}(G \curvearrowright Z) \leq h_{top}(G \curvearrowright Y) + \varepsilon.$$

Proof. Fix $\varepsilon > 0$. By Equation (1) there exists $D \Subset G$ so that $\log(|L_D(Y)| \leq |D|(h_{\text{top}}(G \curvearrowright Y) + \varepsilon)$. Let $\mathcal{F}_1$ be a set of forbidden patterns which defines X and let $\mathcal{F} = \mathcal{F}_1 \cup (\Sigma^D \setminus L_D(Y))$. Letting Z be the G-SFT defined by the set of forbidden patterns $\mathcal{F}$, we have $Z \subset X$ and $Y \subset Z$ from where it follows that $h_{\text{top}}(G \curvearrowright Y) \leq h_{\text{top}}(G \curvearrowright Z)$. Furthermore, by construction we get $L_D(Z) = L_D(Y)$ and thus we have

$$h_{\text{top}}(G \curvearrowright Z) = \inf_{F \in \mathcal{F}(G)} \frac{1}{|F|} \log(|L_F(Z)|) \leq \frac{1}{|D|} \log(|L_D(Z)| \leq h_{\text{top}}(G \curvearrowright Y) + \varepsilon.$$

And so Z satisfies the required properties. □

Let T, K be finite subsets of G. The K-core of T is the set $\text{Core}_K(T) = \{t \in T \mid Kt \subset T\}$. It is an easy exercise to show that if T is a $(K, \frac{\delta}{|K|})$-invariant set, then $|T \setminus \text{Core}_K(T)| < \delta|T|$, for a proof, see [10, Lemma 2.6].

Now we are ready to state the main theorem of this section which shows that every SFT admits subsystems with arbitrarily low topological entropy and which are also SFTs.

Theorem 3.17. *Let G be a countable amenable group and $X \subset \Sigma^G$ be a G-SFT. For every $\varepsilon > 0$ there exists a G-SFT $Z \subset X$ such that $h_{top}(G \curvearrowright Z) \leq \epsilon$*

Proof. We claim that it suffices to show that for every $\varepsilon > 0$ there exists a G-SFT Y (on a different alphabet) such that $h_{top}(G \curvearrowright Y) \leq \epsilon$ and a continuous G-equivariant map $\phi \colon Y \to X$. Indeed, if this is the case, using the above result with $\frac{\varepsilon}{2}$ and the property that (for amenable group actions) topological entropy does not increase under topological factor maps, we obtain that $\phi(Y)$ is a subshift of X with entropy $h_{\text{top}}(G \curvearrowright \phi(Y)) \leq \frac{\varepsilon}{2}$. Using Lemma 3.16 with $\frac{\varepsilon}{2}$ we obtain an SFT $Z \subset X$ whose entropy is bounded by $h_{\text{top}}(G \curvearrowright \phi(Y)) + \frac{\varepsilon}{2} \leq \varepsilon$ as required.

Let us show the above claim. Let $\mathcal{F}$ be a finite set of forbidden patterns which defines X, let $F = \bigcup_{p \in \mathcal{F}} \text{supp}(p)$ be the union of their supports and $K = FF^{-1}$. By Theorem 3.15 there exists a tileset $\mathcal{T} = \{T_1, \ldots, T_n\}$ such that every tile in $\mathcal{T}$ is $(K, \frac{\varepsilon}{4|K|\log(|\Sigma|)})$-invariant and which admits a tiling τ^* by $\mathcal{T}$ with zero entropy. In particular, the K-core of each tile $T \in \mathcal{T}$ satisfies $|T \setminus \text{Core}_K(T)| < \frac{\varepsilon}{4\log(|\Sigma|)}|T|$ and we can find a finite set $D \Subset G$ such that $\log(|L_D(\overline{\{g\tau^*\}_{g \in G}})|) \leq \frac{\varepsilon}{4}|D|$.

By Lemma 3.13 the set $X_{\mathcal{T}}$ of all tilings of G by $\mathcal{T}$ is a G-SFT. Consider the subshift of finite type $X_{\mathcal{T}}^{\mathcal{L}} \subset X_{\mathcal{T}}$ where we additionally forbid the finite set of patterns $\mathcal{L}$:

$$\mathcal{L} = (\mathcal{T} \cup \{\varnothing\})^D \setminus L_D(\overline{\{g\tau^*\}_{g \in G}}).$$

Clearly $\tau^* \in X_{\mathcal{T}}^{\mathcal{L}}$, hence $X_{\mathcal{T}}^{\mathcal{L}}$ is a non-empty G-SFT. Furthermore we have

$$h_{\text{top}}(G \curvearrowright X_{\mathcal{T}}^{\mathcal{L}}) = \inf_{F \Subset G} \frac{1}{|F|} \log(|L_F(X_{\mathcal{T}}^{\mathcal{L}})|) \leq \frac{1}{|D|} \log(|L_D(X_{\mathcal{T}}^{\mathcal{L}})|) \leq \frac{\varepsilon}{4}.$$

Consider the set $U \triangleq \bigcup_{i \leq n} T_i$. We define $X^\star$ as the set of all configurations in $(\Sigma \cup U)^G$ for which no forbidden patterns from $\mathcal{F}$ appear. Finally, we define $Y \subset X_{\mathcal{T}}^{\mathcal{L}} \times X^\star$ as the set of all pairs of configurations (τ, x) such that for every $g \in G$ if we let $(\tau', x') = (g\tau, gx)$ then we have:

1. If $h \in \text{Core}_K(\tau'(1_G))$ then $x'(h) = h$.
2. If $h \in \tau'(1_G) \setminus \text{Core}_K(\tau'(1_G))$ we have $x'(h) \in \Sigma$.
3. $x'|_{\tau'(1_G) \setminus \text{Core}_K(\tau'(1_G))} \in L_{\tau'(1_G) \setminus \text{Core}_K(\tau'(1_G))}(X)$.

In other words, Y is the G-subshift which consists of all configurations obtained by overlaying some $x \in X$ with a tiling $\tau \in X_{\mathcal{T}}^{\mathcal{L}}$ and replacing every symbol in the K-core of a tile by an address pointing to the center of the tile.

We claim Y is a G-SFT. Indeed, it can be obtained from the G-SFT $X_{\mathcal{T}}^{\mathcal{L}} \times X^\star$ by forbidding the finite collection of all patterns p with support U for which the first coordinate of $p(1_G)$ is some $T \in \mathcal{T}$ and either there is $g \in \text{Core}_K(T)$ for which the second coordinate of $p(g)$ is not g or the pattern obtained by restricting the second coordinate of p to $T \setminus \mathrm{Core}_K(T)$ is not in $L_{T\setminus \mathrm{Core}_K(T)}(X)$. We leave it as an exercise to the reader to verify that $(\tau, x) \in Y$ if and only if no patterns as above appear.

Let us first construct the G-equivariant map $\phi\colon Y \to X$. Informally, ϕ is the map that erases the tiling τ and replaces the addresses (which appear in the K-core of some Tg for $T \in \mathcal{T}$) by the symbols of some fixed pattern which depends only on the values of x on $Tg \setminus \mathrm{Core}_K(T)g$. Formally, associate to every $T \in \mathcal{T}$ and pattern $p \in L_{T\setminus \mathrm{Core}_K(T)}(X)$ a pattern $\eta(T,p) \in L_T(X)$ such that $\eta(T,p)|_{T\setminus \mathrm{Core}_K(T)} = p$. Let $\Phi\colon Y \to \Sigma$ be defined by

$$\Phi(\tau, x) \triangleq \begin{cases} x(1_G) & \text{if } x(1_G) \in \Sigma \\ \eta(\tau(h^{-1}), (h^{-1}x)|_{\tau(h^{-1})\setminus \mathrm{Core}_K(\tau(h^{-1}))})(h) & \text{if } x(1_G) = h \in U. \end{cases}$$

As U is finite this map is local. As a consequence, $\phi\colon Y \to \Sigma^G$ given by $\phi(\tau,x)(g) = \Phi(g\tau, gx)$ is a continuous G-equivariant map.

Let us show that $\phi(\tau, x) \in X$. If it is not the case, then there exists $p \in \mathcal{F}$ and $g \in G$ such that $\phi(g\tau, gx)|_{\mathrm{supp}(p)} = p$. For simplicity, let us rename $(\tau', x') = (g\tau, gx)$. If for every $s \in \mathrm{supp}(p)$ we have $x'(s) \in \Sigma$ then $\phi(\tau', x')|_{\mathrm{supp}(p)} = x'|_{\mathrm{supp}(p)}$ which cannot be p by definition of $X^{\star}$. Otherwise we have $\bar{s} \in \mathrm{supp}(p)$ such that $x'(\bar{s}) = h \in U$, which in turn means that $\tau'(h^{-1}\bar{s}) \in \mathcal{T}$. In other words, for $f = h^{-1}\bar{s}$ we have $\bar{s} \in \mathrm{Core}_K(\tau'(f))f$. By definition of K-core, we have that $K\bar{s} \subset \tau'(f)f$. As $\mathrm{supp}(p) \subset F$ and $K = FF^{-1}$ we obtain that $\mathrm{supp}(p) \subset \tau'(f)f$. By definition of ϕ and η we have that $\phi(f\tau', fx')|_{\tau'(f)} = \eta(\tau'(f), fx'|_{\tau'(f)\setminus \mathrm{Core}_K(\tau'(f))}) \in L_{\tau'(f)}(X)$. In particular, $\phi(\tau', x')|_{\tau'(f)f} \in L_{\tau'(f)f}(X)$. As $\mathrm{supp}(p) \subset \tau'(f)f$ this shows that $\phi(\tau', x')|_{\mathrm{supp}(p)} \neq p$, raising a contradiction.

Lastly, let us verify that $h_{\mathrm{top}}(G \curvearrowright Y) \leq \varepsilon$. As $h_{\mathrm{top}}(G \curvearrowright X^{\mathcal{L}}_{\mathcal{T}}) \leq \frac{\varepsilon}{4}$, we can find $W_1 \Subset G$ and $\delta_1 > 0$ such that any (W_1, δ_1)-invariant set R satisfies $\log(|L_R(X^{\mathcal{L}}_{\mathcal{T}})|) \leq |R|\frac{\varepsilon}{2}$. Pick $W \triangleq W_1 \bigcup U$ and $\delta < \delta_1$ sufficiently small (for instance $\delta < \min(\delta_1, \frac{\varepsilon}{4|U|\log(|\Sigma|)})$) such that any (W, δ)-invariant set R satisfies that $|R \setminus \mathrm{Core}_U(R)| < \frac{\varepsilon}{4\log(|\Sigma|)}|R|$.

Fix $\tau \in X^{\mathcal{L}}_{\mathcal{T}}$ and let us denote by $L_R(Y,\tau)$ the set of $p \in L_R(Y)$ for which the first coordinate is $\tau|_R$. Let us write R as the disjoint union $R_1 \uplus R_2$ where R_1 is the set of all $g \in R$ for which there is $h \in R$ such that $\tau(h)h \subset R$. By definition, as $\tau(h) \subset U$, we have that $R_2 \subset R \setminus \mathrm{Core}_U(R)$ and hence $|R_2| < \frac{\varepsilon}{4\log(|\Sigma|}|R|$. On the other hand, the symbols in every position in $\mathrm{Core}_K(\tau(h))h$ are fixed. As the $\tau(h)h$ cover R_1 and $|\tau(h)h \setminus \mathrm{Core}_K(\tau(h)h)| < \frac{\varepsilon}{4\log(|\Sigma|)}|\tau(h)|$ we have at most $\frac{\varepsilon}{4\log(|\Sigma|)}|R_1| \leq \frac{\varepsilon}{4\log(|\Sigma|)}|R|$ positions in R_1 are potentially free. Therefore we obtain the bound

$$|L_R(Y,\tau)| \leq |\Sigma|^{|R_2|}|\Sigma|^{\frac{\varepsilon}{4\log(|\Sigma|)}|R_1|} \leq |\Sigma|^{\frac{\varepsilon}{2\log(|\Sigma|)}|R|}.$$

Note that this does not depend upon the choice of τ. We can thus obtain

$$|L_R(Y)| \leq |L_R(X^{\mathcal{L}}_{\mathcal{T}})||\Sigma|^{\frac{\varepsilon}{2\log(|\Sigma|)}|R|} \leq \exp(|R|\frac{\varepsilon}{2})|\Sigma|^{\frac{\varepsilon}{2\log(|\Sigma|)}|R|}.$$

Therefore

$$h_{\mathrm{top}}(G \curvearrowright Y) \leq \frac{1}{|R|}\log(|L_R(Y)|) \leq \frac{1}{|R|}\left(|R|\frac{\varepsilon}{2} + |R|\frac{\varepsilon\log(|\Sigma|)}{2\log(|\Sigma|)}\right) \leq \varepsilon.$$

Which completes the proof. □

Before applying Theorem 3.17 to reduce the entropy of a chart, let us mention a nice application which shows that for any arbitrary countable amenable group, every subshift of finite type must necessarily contain a subsystem with zero topological entropy. This extends the result of Quas and Trow [22, Corollary 2.3] which shows that minimal $\mathbb{Z}^d$-SFTs have zero topological entropy and whose argument works for amenable orderable groups. Let us also remark the work of Frisch and Tamuz [12] also gives a way to obtain Quas and Trow's result for arbitrarily countable amenable groups and that the author is aware of a non-published direct proof by Ville Salo which works for any amenable and finitely generated group and relies on a combinatorial argument.

Corollary 3.18. *Let be G a countably infinite amenable group. Any G-SFT X contains a G-invariant closed subset with zero topological entropy. In particular, every minimal G-SFT has zero topological entropy.*

Proof. Let $\varepsilon_n = \frac{1}{n}$ and let $Y_0 = X$. By Theorem 3.17 there exists a G-SFT Y_1 such that $h_{\text{top}}(G \curvearrowright Y_1) \leq \varepsilon_1$ and $Y_1 \subset Y_0$. Iterating this procedure we can obtain for every $n \in \mathbb{N}$ a G-SFT Y_n such that $h_{\text{top}}(G \curvearrowright Y_n) \leq \varepsilon_n = \frac{1}{n}$ and $Y_n \subset Y_{n-1}$. As each Y_n is closed we have that $Z = \bigcap_{n \geq 0} Y_n$ is non-empty. Clearly Z is G-invariant as each Y_n is G-invariant. Furthermore, $h_{\text{top}}(G \curvearrowright Z) \leq h_{\text{top}}(G \curvearrowright Y_n)$ for every $n \in \mathbb{N}$, therefore $h_{\text{top}}(G \curvearrowright Z) = 0$. □

Let us also remark that this result is in direct contrast with existence of minimal Toeplitz subshifts of arbitrary positive topological entropy on residually finite groups, see [8, 18, 19].

To the knowledge of the author, the following question is open even in $\mathbb{Z}^2$.

Question 3.19. *Does there exist an amenable group G and a G-SFT which does not contain a zero-entropy G-SFT?*

Let us go back to reducing the entropy of a chart.

Corollary 3.20. *Suppose there exists a free G-chart (X, γ) for H such that X is a G-SFT. Then for every $\varepsilon > 0$ there exists a free G-chart (Y, γ') for H such that Y is a G-SFT and $h_{top}(G \curvearrowright Y) \leq \epsilon$.*

Proof. Apply Theorem 3.17 to X and $\varepsilon > 0$ to obtain a G-SFT Y such that $h_{\text{top}}(G \curvearrowright Y) \leq \epsilon$ and $Y \subset X$. Let $\gamma' \colon H \times Y \to G$ be the restriction of γ to Y. Clearly γ' is continuous and an H-cocycle. □

3.3 Conditions for the existence of free charts

In this section we shall present conditions under which there exist free charts and conditions under which they can be realized with a subshift of finite type. An obvious condition which implies the existence of a free G-chart of H is that H embeds into G as a subgroup, see Example 3.4. Note that in that case the chart automatically has entropy zero and we obtain the rather obvious corollary that $\mathcal{E}_{\text{SFT}}(H) \subset \mathcal{E}_{\text{SFT}}(G)$.

The notion that H embeds into G can be relaxed using the notion of translation-like action introduced by Whyte [26]. We shall see that whenever the groups are finitely generated, this notion is closely related with the existence of free charts.

Definition 3.21. *Let (X, d) be a metric space and H a group. We say that $H \curvearrowright X$ is a* ***translation-like*** *action if*

- *$H \curvearrowright X$ is free, that is, for every $x \in X$ then $hx = x$ implies that $h = 1_H$.*
- *$H \curvearrowright X$ is bounded, that is, for every $h \in H$, $\sup_{x \in X} d(x, hx) < \infty$.*

Any finitely generated group G can be seen as a metric space by endowing it with a metric induced by a finite set of generators. In that case, the second condition can be replaced by the condition that for every fixed $h \in H$ the set of all $(h \cdot g)g^{-1}$ is finite.

Proposition 3.22. *Let H, G be finitely generated groups. H acts translation-like on G if and only if there exists a free G-chart (X, γ) of H.*

Proof. Fix a finite set S of generators of H. Suppose there exists a translation-like action $H \curvearrowright G$. As the action is bounded and S is finite, we have that the set $F = \{f \in G \mid (s \cdot g) = fg \text{ for } s \in S, g \in G\}$ is finite. Consider the alphabet $\Sigma = F^S$ and the configuration $x \colon G \to \Sigma$ such that $(x(g))(s) = f \in F$ if and only if $s \cdot g = fg$. Let $X = \overline{\bigcup_{g \in G}\{gx\}}$ be the orbit closure of x. By definition X is a G-subshift. For $y \in X$, let $\gamma(s, y) = (y(1_G))(s)$ and extend γ to $H \times X$ through the cocycle equation. It is clear that γ is continuous. By definition, we have that $s \cdot_x g = \gamma(gx, s)g = (x(g))(s) = (s \cdot g)g^{-1}g = s \cdot g$. In other words, the action $H \overset{x}{\curvearrowright} G$ coincides with $H \curvearrowright G$ and hence it's free. It follows from compactness that the same holds for any $y \in X$ and thus (X, γ) is a free G-chart of H.

Conversely, suppose there exists a free G-chart (X, γ) of H and let $x \in X$. By definition, the action $H \overset{x}{\curvearrowright} G$ is free. Let $h \in G$, the restriction of γ to $\{h\} \times X$ takes finitely many values and depends only on finitely many coordinates of $x \in X$. It follows that $(h \cdot_x g)g^{-1} = \gamma(h, gx)gg^{-1} = \gamma(h, gx)$ takes only finitely many values and hence $H \overset{x}{\curvearrowright} G$ is bounded. □

In other words, the least we can require if we want a free G-chart of H is the existence of a translation-like action of H on G. In what follows we shall give further conditions under which one can always find a free G-chart of H given by a G-SFT. The following proof is essentially contained in the work of Jeandel [15, Section 2].

Proposition 3.23. *Let H, G be finitely generated groups such that:*

1. *H admits a translation-like action on G.*
2. *H is finitely presented.*
3. *There exists a non-empty H-SFT for which the H-action is free.*

Then there exists a free G-chart (X, γ) of H such that X is a non-empty G-SFT.

Proof. The first part of the proof is the same as in the last proposition, let $H \curvearrowright G$ be the translation-like action and suppose $\langle S \mid R \subset S^* \rangle$ is a finite presentation of H where $S = S^{-1}$. By definition, the set $F = \{f \in G \mid (s \cdot g) = fg \text{ for } s \in S, g \in G\}$ is finite. Consider the alphabet $\Sigma = F^S$ of all functions from S to F and let $\gamma\colon S^* \times \Sigma^G \to G$ be the map given by $\gamma(s, x) = (x(1_G))(s)$ for $s \in S$ and extended to the free monoid S^* by the condition

$$\gamma(s_1 s_2, x) = \gamma(s_1, \gamma(s_2, x)x) \cdot \gamma(s_2, x) \text{ for every } s_1, s_2 \text{ in } S^*.$$

Let us first consider the subshift $Y \subset \Sigma^G$ such that for every $s \in S$ and $g \in G$ we have $(y(g))(s) = f$ then $(y(fg))(s^{-1}) = f^{-1}$. This is clearly a subshift of finite type. Let us note that for $y \in Y$, $g \in G$ and $s \in S$ we have,

$$\begin{aligned}\gamma(s^{-1}s, gy) &= \gamma(s^{-1}, \gamma(s, gy)gy) \cdot \gamma(s, gy)\\ &= \gamma(s^{-1}, [(y(g))(s)]gy) \cdot (y(g))(s)\\ &= y([(y(g))(s)](s^{-1}) \cdot (y(g))(s) = 1_G.\end{aligned}$$

The same holds for $\gamma(ss^{-1}, gy)$. By a similar argument, it can be shown that if $w \in S^*$ is a word that can be freely reduced to the identity, then $\gamma(w, gy) = 1_G$ for every $g \in G$. In other words, (Y, γ) codes the free group on S generators.

Let us define $X \subset Y$ as the set of all configurations $x \in Y$ such that whenever $s_1 s_2 \dots s_{n-1} s_n \in R$ then for every $g \in G$, if we define $f_1 = (y(g))(s_n)$, $f_2 = (y(f_1 g))(s_{n-1})$ and for every $k \leq n$,

$$f_k = (y(f_{k-1} \dots f_1 g))(s_{n+1-k}).$$

Then we have $f_n f_{n-1} \dots f_1 = 1_G$. As R is finite, these conditions can be imposed by forbidding patterns with support bounded by F^n. In other words, X is also a G-subshift of finite type. Again, by the previous calculation, we obtain that for every $w \in R$ and $g \in G$ we have $\gamma(w, gx) = 1_G$ Moreover, as every word which represents 1_G in G can be obtained by freely conjugating and concatenating words in R, we have that any word $w \in S^*$ which represents the identity satisfies $\gamma(w, gx) = 1_G$. In other words, (X, γ) codes a G-chart of H.

It is not true that (X, γ) is free. In fact, the configuration such that $(x(g))(s) = 1_G$ belongs to X. However, the configuration $\bar{x} \in X$ defined using the free action $H \curvearrowright G$ by $(\bar{x}(g))(s) = (s \cdot g)g^{-1}$ satisfies that $H \overset{x}{\curvearrowright} G = H \curvearrowright G$.

By hypothesis, there exists an H-subshift Z on which H acts freely. Let us consider $Z_\gamma[X]$. By Remark 3.9 we have that $Z_\gamma[X]$ is non-empty. Let $\widehat{\gamma}\colon H \times Z_\gamma[X] \to G$ be the map defined by $\widehat{\gamma}(h, (z, x)) = \gamma(h, x)$. We claim the G-chart $(Z_\gamma[X], \widehat{\gamma})$ of H is free.

Indeed, if it is not free, there is $(z, x) \in Z_\gamma[X]$ and $h \neq 1_H$ such that $h \cdot_{(z,x)} g = g$. Equivalently, such that $\gamma(h, gx) = 1_G$ or $h \cdot_x g = g$. Hence, we would have that

$$h\pi_{x,g}(z) = \pi_{x, h \cdot_x g}(z) = \pi_{x,g}(z).$$

As $\pi_{x,g}(z) \in Z$, this gives a configuration for which the shift does not act freely, which contradicts the assumption on Z. □

Let us gather all our results in a single theorem for further reference.

Theorem 3.24. *Let G, H be finitely generated amenable groups. Suppose that*

1. *H admits a translation-like action on G.*
2. *H is finitely presented.*

3. There exists a non-empty H-SFT for which the H-action is free.

Then, for every $\varepsilon > 0$ there exists a G-SFT X such that $h_{top}(G \curvearrowright X) < \varepsilon$ and

$$h_{top}(G \curvearrowright X) + \mathcal{E}_{SFT}(H) \subset \mathcal{E}_{SFT}(G).$$

Proof. By Proposition 3.23 there exists a free G-chart (X, γ) of H such that X is a G-SFT. Furthermore, by Corollary 3.20 we can choose it so that $h_{top}(G \curvearrowright X) < \varepsilon$. Finally, we conclude by applying Corollary 3.11. □

4 Characterization of entropies: the case $H = \mathbb{Z}^2$

The goal of this section is to exploit Theorem 3.24 for the case $H = \mathbb{Z}^2$. The interest on this particular case comes from the fact we already have a full characterization of the entropies of $\mathbb{Z}^2$-SFTs by Theorem 2.7. Furthermore, $\mathbb{Z}^2 \cong \langle a, b \mid aba^{-1}b^{-1} \rangle$ is finitely presented, and there exist non-empty $\mathbb{Z}^2$-SFTs for which the $\mathbb{Z}^2$-action is free, for instance the Robinson tiling [23].

There is a single obstacle that stops us from getting a characterization for all groups on which $\mathbb{Z}^2$ acts translation-like: even if we can choose the entropy of the chart to be arbitrarily low, there is no guarantee that said entropy will be an upper semi-computable number. In what follows we shall show that this is indeed the case if G is a finitely generated group with decidable word problem.

Given a set $S \subset G$ denote by S^* the formal set of all finite words $s_1 s_2 \dots s_n \in S^*$. Also, for any such word in S^* denote by $\underline{s_1 s_2 \dots s_n}$ the unique element of G represented by it.

Definition 4.1. *Let G be a finitely generated group and S a finite set of generators. The* ***word problem*** *of G is the set of all words over the alphabet S which represent the identity of G.*

$$\texttt{WP}_S(G) = \{w \in S^* \mid \underline{w} = 1_G\}.$$

We say that G has **decidable word problem** if the language $\texttt{WP}_S(G)$ is decidable for some finite set of generators S. It can be shown that this notion is independent of the chosen set of generators and thus, modulo many-one equivalence, one can speak about the **word problem** $\texttt{WP}(G)$ of G without making reference to a specific set of generators.

We shall also need to introduce the set of locally admissible patterns.

Definition 4.2. *Let Σ be a finite alphabet and $\mathcal{F}$ be a list of forbidden patterns which defines a subshift $X_{\mathcal{F}}$. For $F \Subset G$ We say that $q \in \Sigma^F$ is in the set of* ***locally admissible patterns*** *$L_F^{\texttt{loc}}(X_{\mathcal{F}})$ if no patterns from $\mathcal{F}$ appear in q, namely, $[q] \not\subset g[p]$ for every $g \in G$ and $p \in \mathcal{F}$.*

Lemma 4.3. *Let G be a countable group and $X_{\mathcal{F}} \subset \Sigma^G$ be a subshift defined by a set of forbidden patterns $\mathcal{F}$. For any $F \Subset G$ there exists $K \Subset G$ such that $K \supset F$ and $p \in L_F(X)$ if and only if there exists $q \in L_K^{\texttt{loc}}(X)$ such that $q|_F = p$.*

Proof. If G is finite the result is obvious. Otherwise we may fix an enumeration $\{g_n\}_{n \in \mathbb{N}}$ of G, let $F^n = F \cup \bigcup_{k \leq n} \{g_k\}$ and consider $p \in \Sigma^F \setminus L_F(X)$. We claim there must exist an integer $n(p)$ such that $q|_F \neq p$ for every $q \in L_{F^{n(p)}}^{\texttt{loc}}(X)$. If this was not the case, we may choose for every n a pattern $q^n \in L_{F^n}^{\texttt{loc}}(X)$ such that $q|_F = p$. As the sequence of $[q_n] \subset [p]$ is closed and nested, the intersection $Y = \bigcap_{n \in \mathbb{N}} [q_n]$ is non-empty and $Y \subset [p]$, and any configuration $y \in Y$ satisfies that no forbidden patterns appear, hence $y \in X \cap [p]$ and thus $p \in L_F(X)$.

As Σ^F is finite we may define $N \triangleq \max_{p \in \Sigma^F \setminus L_F(X)} n(p)$ and $K \triangleq F^N$. By definition of N, we have that if $p \in \Sigma^F \setminus L_F(X)$ then $q|_F \neq p$ for every $q \in L_K^{\texttt{loc}}(X)$. Conversely, if $p \in L_F(x)$ there exists x such that $x|_F = p$. Defining $q \triangleq x|_K$ we have $q|_F = p$ and $q \in L_K(X) \subset L_K^{\texttt{loc}}(X)$. □

In what follows we shall need to briefly introduce the notion of pattern codings and effectively closed subshifts in finitely generated groups. An introduction to this topic can be found on [3].

Definition 4.4. *Let G be a finitely generated group, S a finite set of generators and Σ an alphabet. A function $c \colon W \to \Sigma$ from a finite subset W of S^* is called a* ***pattern coding****. The cylinder defined by a pattern coding c is given by*

$$[c] = \bigcap_{w \in W} \underline{w}[c(w)].$$

In other words, a pattern coding is a coloring of a finite subset of the free monoid S^*. A set $\mathcal{C}$ of pattern codings defines a G-subshift $X_{\mathcal{C}}$ by setting

$$X_{\mathcal{C}} = \Sigma^G \setminus \bigcup_{g \in G, c \in \mathcal{C}} g[c].$$

We say that a G-subshift X is **effectively closed** if there exists a recursively enumerable set of pattern codings $\mathcal{C}$ such that $X = X_{\mathcal{C}}$. Obviously, every G-SFT is effectively closed.

We shall need the following result.

Lemma 4.5 (Lemma 2.3 of [3]). *Let G be a finitely generated and recursively presented group. For every effectively closed subshift $X \subset \Sigma^G$ the maximal –for inclusion– set of forbidden pattern codings that defines X is recursively enumerable.*

Proposition 4.6. *Let G be a finitely generated amenable group with decidable word problem. For every effectively closed subshift $X \subset \Sigma^G$ the topological entropy $h_{top}(G \curvearrowright X)$ is upper semi-computable.*

Proof. Let us fix a symmetric set S of generators for G. We shall first define three algorithms $T_{\mathtt{WP}}, T_{\mathtt{pat}}, T_{\mathtt{color}}$ which will be used in the proof.

First, as G has decidable word problem there is an algorithm $T_{\mathtt{WP}}$ which on input $w \in S^*$ halts and accepts if and only if $\underline{w} = 1_G$.

Second, as X is effectively closed, by Lemma 4.5 there exists a maximal recursively enumerable set of pattern codings $\mathcal{C}^*$ such that $X = X_{\mathcal{C}^*}$. We define $T_{\mathtt{pat}}$ as the algorithm which on input $n \in \mathbb{N}$ yields the list of the first n pattern codings $[c_1, c_2, \dots, c_n]$ of $\mathcal{C}^*$.

Finally, let us denote by $\equiv_n$ the equivalence relation on $\bigcup_{k \leq n} S^k$ defined by $u \equiv_n v$ if and only if $T_{\mathtt{WP}}$ accepts uv^{-1}. Let $B_n \triangleq \bigcup_{k \leq n} S^k / \equiv_n$. We define $T_{\mathtt{color}}$ as the algorithm, which on input $n \in \mathbb{N}$ computes the set of all functions $x \colon B_n \to \Sigma$ such that for every pattern coding $c_i \colon W_i \to \Sigma$ listed by $T_{\mathtt{pat}}$ on input n we have that either $W_i \setminus B_n \neq \emptyset$ or $x(w) \neq c(w)$ for at least one $w \in W_i$.

In simpler words, $T_{\mathtt{color}}$ enumerates all patterns over a representation of the ball of size n of the Cayley graph of G where the first n forbidden pattern codings do not appear at the identity.

Now we construct an algorithm $T_{\mathtt{ent}}$ which on input n outputs a rational number h_n as follows. First apply algorithm $T_{\mathtt{color}}$ on input n to produce a set $\{x_1, \dots, x_{M(n)}\}$ of colorings as above. For each $A \subset B_n$ we define L_n^A as the set of restrictions $\{x_1|_A, \dots, x_{M(n)}|_A\}$ to A. Let us define h_n^A as the smallest rational number of the form $\frac{k}{2^n}$ such that

$$\frac{1}{|A|} \log(|L_n^A|) < \frac{k}{2^n}.$$

Finally, let us define $h_n \triangleq \min_{A \subset B_n}\{h_n^A\}$. From the above definitions, it is clear that each h_n can be computed in a finite number of steps with $T_{\mathtt{ent}}$. We claim that the sequence $\{h_n\}_{n \in \mathbb{N}}$ is non-increasing and that $\inf_{n \in \mathbb{N}} h_n = h_{\text{top}}(G \curvearrowright X)$.

Indeed, let $m > n$. Clearly for $A \subset B_n$ we have $L_m^A \subset L_n^A$, hence $|L_n^A| > |L_m^A|$ hence we obtain

$$h_m = \min_{A \subset B_m}\{h_m^A\} \leq \min_{A \subset B_n}\{h_m^A\} \leq \min_{A \subset B_n}\{h_n^A\} = h_n.$$

Hence the sequence $\{h_n\}_{n \in \mathbb{N}}$ is non-increasing. It is clear from the definition that for every $n \in \mathbb{N}$ such that $B_n \supset A$ we have $L_n^A \supset L_A(X)$, hence $h_n^A > \frac{1}{|A|}\log(|L_n^A|) \geq \frac{1}{|A|}\log(|L^A(X)|)$ and thus by Equation (1),

$$h_n > \inf_{A \subset B_n} \frac{1}{|A|} \log(|L_A(X)|) \geq h_{\text{top}}(G \curvearrowright X).$$

Similarly, by Equation (1) for every $\varepsilon > 0$ there exists a fixed finite $F \subset G$ such that $\frac{1}{|F|}\log(|L_F(X)|) - h_{\text{top}}(G \curvearrowright X) \leq \epsilon$. By Lemma 4.3 there exists K such that $p \in L_F(X)$ if and only if there exists $q \in L_K^{\mathtt{loc}}(X)$ such that $q|_F = p$. Choose N_1 such that $B_{N_1} \supset K$ and N_2 so that all pattern codings of $\mathcal{C}^*$ whose support is contained in K have already appeared. Let $N \geq \max(N_1, N_2)$. By definition we have that $L_N^K = L_K^{\mathtt{loc}}(X)$ and thus $L_N^F = L_F(X)$, hence we have that

$$h_N \leq h_N^F \leq \frac{1}{|F|}\log(|L_F(X)|) + \frac{1}{2^N} \leq h_{\text{top}}(G \curvearrowright X) + \epsilon + \frac{1}{2^N}.$$

The last inequality shows that $\{h_n\}_{n \in \mathbb{N}}$ converges to $h_{\text{top}}(G \curvearrowright X)$. □

From this, we can obtain the following characterization.

Theorem 4.7. *Let G be a finitely generated amenable group with decidable word problem which admits a translation-like action by $\mathbb{Z}^2$. The set of entropies attainable by G-subshifts of finite type is the set of non-negative upper semi-computable numbers.*

Proof. By hypothesis there exists a translation-like action of $\mathbb{Z}^2$ on G. Therefore $\mathbb{Z}^2, G$ satisfy the hypothesis of Theorem 3.24 which means that for every $\varepsilon > 0$ there exists a G-SFT X such that $h_{top}(G \curvearrowright X) < \varepsilon$ and

$$h_{top}(G \curvearrowright X) + \mathcal{E}_{\mathrm{SFT}}(\mathbb{Z}^2) \subset \mathcal{E}_{\mathrm{SFT}}(G).$$

Recall that by Theorem 2.7 $\mathcal{E}_{\mathrm{SFT}}(\mathbb{Z}^2)$ is precisely the set of non-negative upper semi-computable real numbers. As G has decidable word problem, Proposition 4.6 implies that $\mathcal{E}_{\mathrm{SFT}}(G) \subset \mathcal{E}_{\mathrm{SFT}}(\mathbb{Z}^2)$. Noting that $0 \in \mathcal{E}_{\mathrm{SFT}}(G)$ and that the set of upper semi-computable numbers is stable under addition, if we let ε go to zero we obtain

$$\mathcal{E}_{\mathrm{SFT}}(G) = \mathcal{E}_{\mathrm{SFT}}(\mathbb{Z}^2).$$

Which is what we wanted to show □

5 Consequences

In the remainder of this section we shall make use of the following simple construction.

Definition 5.1. *Let $H \leq G$ be a subgroup, $\{0\}$ be the trivial G-subshift with one point and let the H-cocycle $\gamma\colon H \times \{0\} \to G$ be the canonical free G-chart of H defined by $\gamma(h, 0) = h$.*

*For an H-subshift X denote by $X^{\uparrow G}$ the **free G-extension of** X defined by $X_\gamma[\{0\}]$.*

Proposition 5.2. *Let G be a countable amenable group, $H \leq G$ and X be an H-subshift. Then*

$$h_{top}(G \curvearrowright X^{\uparrow G}) = h_{top}(H \curvearrowright X).$$

Proof. By Theorem 3.10 we have

$$h_{\mathrm{top}}(G \curvearrowright X^{\uparrow G}) = h_{\mathrm{top}}(H \curvearrowright X) + h_{\mathrm{top}}(G \curvearrowright \{0\}) = h_{\mathrm{top}}(H \curvearrowright X).$$

Which is what we wanted to show. □

We shall also need the following result which relates the entropies of subshifts of finite type in a group to those of a finite index subgroup.

Lemma 5.3. *Let G be a countable amenable group and let $H \leq G$ be a finite index subgroup. Assume that $\mathcal{E}_{SFT}(H)$ is closed under division by positive integers. Then $\mathcal{E}_{SFT}(G) = \mathcal{E}_{SFT}(H)$.*

Proof. For any H-SFT X we can consider the G-SFT $X^{\uparrow}$. By Proposition 5.2 we get $\mathcal{E}_{\mathrm{SFT}}(H) \subset \mathcal{E}_{\mathrm{SFT}}(G)$.

For the converse, let $Y \subset \Sigma^G$ be a G-SFT and consider $H \curvearrowright Y$ the restriction of the G action on Y to H. It is a well known property of topological entropy that $\frac{1}{[G:H]} h_{\mathrm{top}}(H \curvearrowright Y) = h_{\mathrm{top}}(G \curvearrowright Y)$. It suffices to show that $H \curvearrowright Y$ is conjugated to an H-SFT. Indeed, as $\mathcal{E}_{\mathrm{SFT}}(H)$ is closed under division by positive integers the above formula yields the result.

Choose a set R of left representatives of G/H and define the R-higher power shift $X^{[R]}$ by

$$X^{[R]} = \{x \in (\Sigma^R)^H \mid \exists y \in Y, \text{ for every } r \in R, h \in H, (x(h))(r) = y(rh)\}.$$

As R is finite, it is clear that $X^{[R]}$ is closed and H-invariant and hence that it is an H-subshift. The function $\phi\colon X^{[R]} \to Y$ that sends $x \mapsto y$ by $\phi(x)(rh) = (x(h))(r)$ is clearly a continuous bijection. It is also H-equivariant:

$$h'\phi(x)(rh) = \phi(x)(rhh') = (x(hh'))(r) = (h'x(h))(r) = \phi(h'x)(rh).$$

Therefore $H \curvearrowright X^{[R]}$ is conjugated to $H \curvearrowright Y$. The construction of the forbidden patterns that show that $X^{[R]}$ is an H-SFT whenever Y is a G-SFT is a simple exercise. The reader may find it in either [7, Definition 3.1] or in [2, Proposition 9.3.33]. □

Question 5.4. *Is there any infinite and finitely generated amenable group G for which $\mathcal{E}_{SFT}(G)$ is not closed under division by positive integers?*

5.1 Polycyclic-by-finite groups

The goal of this section is to give a full characterization of the set of real numbers attainable as entropies of subshifts of finite type on a polycyclic-by-finite group. In what follows we shall introduce polycyclic groups and state a few of their properties. A good reference is [24] or [11].

A group G is called **polycyclic** if there exists a finite sequence of subgroups

$$G = N_1 \rhd N_2 \rhd \cdots \rhd N_n \rhd N_{n+1} = \{1_G\}.$$

such that every quotient N_i/N_{i+1} is cyclic. The number of i such that N_i/N_{i+1} is infinite does not depend on the choice of sequence and is thus a group invariant called the **Hirsch index** of G and denoted by $h(G)$.

If we replace the condition that each N_i/N_{i+1} is cyclic by the condition that each N_i/N_{i+1} is the infinite cyclic group, we obtain the class of **poly-C_∞** groups. There are polycyclic groups which are not poly-C_∞, for instance any cyclic finite group. However, they are very close in the following sense. A proof can be found in either of the two references mentioned above.

Proposition 5.5. *The following are equivalent:*

1. *G is virtually polycyclic.*
2. *G is polycyclic-by-finite.*
3. *G is poly-C_∞-by-finite.*

In particular, as every short sequence $1 \to N \to G \to \mathbb{Z} \to 1$ splits, the last proposition means that any virtually polycyclic group can be written as a series $G = N_0 \rhd N_1 \rhd \cdots \rhd N_n \rhd N_{n+1} = \{1_G\}$ such that for $i \geq 1$ we have $N_i = N_{i+1} \rtimes \mathbb{Z}$ and G is virtually N_1. Moreover if this is the case then $h(G) = n$.

Theorem 5.6. *Let G be a virtually polycyclic group. Then*

1. *If $h(G) = 0$ then $\mathcal{E}_{SFT}(G) = \{\frac{1}{|G|}\log(n) \mid n \in \mathbb{Z}_+\}$.*
2. *If $h(G) = 1$ then $\mathcal{E}_{SFT}(G) = \mathcal{E}_{SFT}(\mathbb{Z})$, the set of non-negative rational multiples of logarithms of Perron eigenvalues.*
3. *If $h(G) \geq 2$ then $\mathcal{E}_{SFT}(G) = \mathcal{E}_{SFT}(\mathbb{Z}^2)$, the set of non-negative upper semi-computable numbers.*

Proof. As G is poly-C_∞-by-finite, we have that $G = N_0 \rhd N_1 \rhd \cdots \rhd N_{h(G)} \rhd N_{h(G)+1} = \{1_G\}$ where every quotient except the first one is an infinite cyclic group. If $h(G) = 0$, then $G = N_0 \rhd N_1 = \{1_G\}$ is necessarily a finite group F. As every Følner sequence in a finite group is eventually the whole group, we have that for any subshift $X \subset \Sigma^F$,

$$h_{\text{top}}(F \curvearrowright X) = \frac{1}{|F|}\log(|L_F(X)|).$$

In particular, the entropy of every subshift is of the form we claim. To show that every such number occurs, consider the SFT $X^n_{\text{unif}} \subset \{1, 2, \dots, n\}^F$ consisting of the uniform configurations x_i such that $x_i(f) = i$ for every $f \in F$. Clearly $h_{\text{top}}(F \curvearrowright X^n_{\text{unif}}) = \frac{1}{|F|}\log(n)$. This proves the first claim.

If $h(G) = 1$ then $G = N_0 \rhd N_1 \rhd N_2 = \{1_G\}$. As $N_1 \cong \{1_G\} \rtimes \mathbb{Z}$ then $N_1 \cong \mathbb{Z}$. This means that G is virtually $\mathbb{Z}$. By Lemma 5.3 the claim holds for this case as well.

Let $h(G) \geq 2$. We will show that $\mathbb{Z}^2$ embeds into G. Indeed, we have that $N_{h(G)} \cong \mathbb{Z}$ and that $N_{h(G)-1} \cong N_{h(G)} \rtimes \mathbb{Z}$ is a subgroup of G. Hence, have that $N_{h(G)-1} \cong \mathbb{Z} \rtimes_\varphi \mathbb{Z}$ for some homomorphism $\varphi\colon \mathbb{Z} \to \text{Aut}(\mathbb{Z})$. There are two cases: either $\varphi(1) = \text{id}$ or $\varphi(1)$ is multiplication by -1. The first case yields $N_{h(G)-1} \cong \mathbb{Z}^2$ and hence $\mathbb{Z}^2$ embeds into G. In the second case note that $\varphi(2) = \text{id}$ and thus $\mathbb{Z} \rtimes_\varphi 2\mathbb{Z}$ is isomorphic to $\mathbb{Z}^2$. Hence $N_{h(G)-1}$ contains a finite index copy of $\mathbb{Z}^2$ and thus as $N_{h(G)-1}$ embeds into G, we obtain that $\mathbb{Z}^2$ embeds into G as well.

Therefore, whenever $h(G) \geq 2$ we have that $\mathbb{Z}^2$ embeds into G. In particular $\mathbb{Z}^2$ acts translation-like on G. As every polycyclic-by-finite group is finitely generated and has decidable word problem, we can apply Theorem 4.7 to obtain the desired conclusion. □

Remark 5.7. In the previous proof we did not use the full power of Theorem 4.7. We only applied it to the case where $\mathbb{Z}^2$ actually embeds into G. The next application will rely strongly on translation-like actions.

5.2 Products of infinite finitely generated groups

In this section we shall make use of the following theorem by Seward [25]

Theorem 5.8 (Theorem 1.4 of [25]). *Every infinite and finitely generated group admits a translation-like action of* $\mathbb{Z}$.

Corollary 5.9. *Let* G_1, G_2 *be infinite and finitely generated groups. Then* $G_1 \times G_2$ *admits a translation-like action of* $\mathbb{Z}^2$.

Proof. By Theorem 5.8, there exist translation-like actions $\mathbb{Z} \overset{\alpha_1}{\curvearrowright} G_1$ and $\mathbb{Z} \overset{\alpha_2}{\curvearrowright} G_2$. The $\mathbb{Z}^2$-action given by $(n_1, n_2) \cdot (g_1, g_2) \triangleq (n_1 \cdot_{\alpha_1} g_1, n_2 \cdot_{\alpha_2} g_2)$ satisfies the requirements. □

Corollary 5.10. *Let* G_1, G_2 *be two infinite, amenable and finitely generated groups with decidable word problem. The set of topological entropies of non-empty* $G_1 \times G_2$*-SFTs is exactly the set of non-negative upper semi-computable numbers.*

Proof. Clearly $G_1 \times G_2$ has decidable word problem. By the previous corollary it admits a translation-like action of $\mathbb{Z}^2$. The result follows from Theorem 4.7. □

5.3 Countably infinite amenable groups

Let us now consider the case of countably infinite amenable groups which are not necessarily finitely generated. In the remainder of this section we will need to speak about the word problem for arbitrary countable groups. We shall say that a group presentation $\langle \mathbb{N} \mid R \subset \mathbb{N}^* \rangle$ has decidable word problem if there exists an algorithm which on entry $w \in \mathbb{N}^*$ decides whether $\underline{w} = 1$ in the group defined by that presentation. We shall say that a countable group G has **decidable word problem** if it admits a presentation with decidable word problem. Note that if G has decidable word problem, then every finitely generated subgroup of G also does, but the converse may not hold, see for instance [4, Example 5.4].

Proposition 5.11. *Let* G *be a countably infinite amenable group which admits a decidable presentation and let* $X \subset \Sigma^G$ *be a* G*-subshift of finite type. Then* $h_{top}(G \curvearrowright X)$ *is upper-semi computable.*

Proof. If X is a G-subshift of finite type, there is a finite set of patterns $\mathcal{F}$ which defines it. Let $S = \bigcup_{p \in \mathcal{F}} \operatorname{supp}(p)$ be the union of the supports of patterns in $\mathcal{F}$ and let $H = \langle S \rangle \leq G$ be the finitely generated subgroup of G generated by S. As G is amenable and has decidable word problem, then H is amenable and has decidable word problem. Let Y be the H-subshift defined by $\mathcal{F}$. We clearly have that $X = Y^{\uparrow G}$ where $Y^{\uparrow G}$ is the free G-extension of Y. By Proposition 4.6 $h_{\text{top}}(H \curvearrowright Y)$ is upper semi-computable. Therefore by Proposition 5.2 we have that $h_{\text{top}}(H \curvearrowright Y) = h_{\text{top}}(G \curvearrowright Y^{\uparrow G}) = h_{\text{top}}(G \curvearrowright X)$ and hence $h_{\text{top}}(G \curvearrowright X)$ is also upper semi-computable. □

Corollary 5.12. *Let* G *be an amenable countably infinite group with decidable word problem and which admits a finitely generated subgroup on which* $\mathbb{Z}^2$ *acts translation-like. Then*

$$\mathcal{E}_{SFT}(G) = \mathcal{E}_{SFT}(\mathbb{Z}^2).$$

Proof. By Proposition 5.11 we get $\mathcal{E}_{\text{SFT}}(G) \subset \mathcal{E}_{\text{SFT}}(\mathbb{Z}^2)$. Let H be a finitely generated subgroup on which $\mathbb{Z}^2$ acts translation-like. As H has decidable word problem and is amenable, by Theorem 4.7 $\mathcal{E}_{\text{SFT}}(H) = \mathcal{E}_{\text{SFT}}(\mathbb{Z}^2)$. For any $r \in \mathcal{E}_{\text{SFT}}(H)$, there is an H-SFT X such that $h_{\text{top}}(H \curvearrowright X) = r$. By Proposition 5.2 we have $h_{\text{top}}(G \curvearrowright X^{\uparrow}) = r$ and hence $\mathcal{E}_{\text{SFT}}(H) \subset \mathcal{E}_{\text{SFT}}(G)$. This gives $\mathcal{E}_{\text{SFT}}(G) = \mathcal{E}_{\text{SFT}}(\mathbb{Z}^2)$. □

Corollary 5.13. *Let* G_1, G_2 *be amenable, countably infinite and non-locally finite groups with decidable word problem. Then*

$$\mathcal{E}_{SFT}(G_1 \times G_2) = \mathcal{E}_{SFT}(\mathbb{Z}^2).$$

Proof. $G_1 \times G_2$ is amenable, countably infinite and has decidable word problem. Furthermore, as neither group is locally finite, there are infinite and finitely generated subgroups $H_1 \leq G_1$ and $H_2 \leq G_2$. By Corollary 5.9 $H_1 \times H_2$ admits a translation-like action of $\mathbb{Z}^2$. The result follows from Corollary 5.12. □

Remark 5.14. The non-locally finite condition in Corollary 5.13 is necessary. If G is a locally finite group and $X \subset \Sigma^G$ is a subshift of finite type. We can use the same technique as in Proposition 5.11 to reduce its entropy to the entropy of the group which is finitely generated by the support of its forbidden patterns. But the entropy of any subshift in a finite group is necessarily a rational multiple of the logarithm of a positive integer.

5.4 Branch groups

Suppose that G is a countable amenable group with decidable word problem which contains the product of two non-locally finite and countably infinite subgroups $G_1 \times G_2$ as a subgroup. Then Corollary 5.13 and Corollary 5.12 imply that $\mathcal{E}_{\mathrm{SFT}}(G) = \mathcal{E}_{\mathrm{SFT}}(\mathbb{Z}^2)$.

There are many examples satisfying the previous hypothesis within the class of branch groups [5]. There is more than one definition of branch group, we shall work with the following one:

Definition 5.15. *A group G is called a **branch group** if there exist two sequences of groups $(L_i)_{i\in\mathbb{N}}$ and $(H_i)_{i\in\mathbb{N}}$ and a sequence of positive integers $(k_i)_{i\in\mathbb{N}}$ such that $k_0 = 1$, $G = L_0 = H_0$ and:*

1. *$\bigcap_{i\in\mathbb{N}} H_i = 1_G$.*
2. *H_i is normal in G and has finite index.*
3. *there are subgroups $L_i^{(1)}, \dots, L_i^{k(i)}$ of G such that $H_i = L_i^{(1)} \times \cdots \times L_i^{k(i)}$ and each of the $L_i^{(j)}$ is isomorphic to L_i.*
4. *Conjugation by elements of g transitively permutes the factors in the above product decomposition.*
5. *k_i properly divides k_{i+1} and each of the factors $L_i^{(j)}$ contains k_{i+1}/k_i factors $L_{i+1}^{(j')}$.*

This allows us to state the following result

Theorem 5.16. *Let G be an infinite, finitely generated, amenable branch group with decidable word problem. Then $\mathcal{E}_{SFT}(G) = \mathcal{E}_{SFT}(\mathbb{Z}^2)$.*

Proof. By the fifth property above, $k_1 > 1$. Furthermore, as each H_i has finite index, it is also infinite and finitely generated. As k_1 is finite, each L_i is also infinite and finitely generated. Thus $H_1 = L_1^{(1)} \times \cdots \times L_1^{(k_1)}$ is a subgroup of G on which $\mathbb{Z}^2$ acts translation-like. The result follows from Corollary 5.12. □

A canonical example which satisfies all of the above properties is the following.

Example 5.17. The set of topological entropies of non-empty SFTs in the Grigorchuk group [13] is exactly the set of non-negative upper semi-computable numbers.

6 Final remarks

The techniques presented in this work give tools to embed the entropies of SFTs defined on a group G to groups in which G embeds geometrically. As the only known non-trivial base cases are $\mathbb{Z}$ and $\mathbb{Z}^2$, we can only obtain characterizations which coincide either with $\mathcal{E}_{\mathrm{SFT}}(\mathbb{Z})$ or $\mathcal{E}_{\mathrm{SFT}}(\mathbb{Z}^2)$. This raises the following question.

Question 6.1. *Is there any infinite and finitely generated amenable group G with decidable word problem for which $\mathcal{E}_{SFT}(G)$ is neither $\mathcal{E}_{SFT}(\mathbb{Z})$ nor $\mathcal{E}_{SFT}(\mathbb{Z}^2)$?*

Furthermore, Theorem 4.7 provides a full characterization of the entropies attainable by SFTs defined on polycyclic-by-finite groups, but it cannot be applied on every solvable group with decidable word problem. Two notable examples where it does not apply (at least not directly) are the Baumslag-Solitar groups $\mathtt{BS}(1,n) = \langle a, b \mid bab^{-1} = a^n \rangle$ for $n \geq 2$, and the Lamplighter group $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$.

Question 6.2. *For $n \geq 2$, does it hold that $\mathcal{E}_{SFT}(\mathtt{BS}(1,n)) = \mathcal{E}_{SFT}(\mathbb{Z}^2)$?*

Question 6.3. *Characterize $\mathcal{E}_{SFT}(\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z})$. Does it coincide with either $\mathcal{E}_{SFT}(\mathbb{Z})$ or $\mathcal{E}_{SFT}(\mathbb{Z}^2)$?*

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